Eigenfunction Asymptotics

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1 Opening Remarks

These are notes taken when attending the three-week semiclassical analysis summer school at Northwestern University, as part of SNAP ("Summer Northwestern Analysis Program"). These notes comprise the first week on eigenfunctions taught by Dr. Yaiza Canzani. Each section represents a lecture. It is worth pointing out that this material is similar Chapter 14 of Zworski's *Semiclassical Analysis* (in particular, see §14.3). However, they work with Schrödinger operators that possess a smooth, real-valued potential V, and we will assume that $V \equiv 0$ (although adding in such potentials does not add significant difficulty).

Any mistakes are my own (especially in places where I filled in extra info). Shoot me an email if/when you catch them!

2 Basic Geometry

In $\mathbb{R}^n$, we define the Lapacian

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} : C^\infty(\Omega) \to C^\infty(\Omega),$$

where $\Omega \subset \mathbb{R}^n$ is compact. Such an operator comes up in the study of

$$\begin{aligned} \text{Heat}: &\quad (\Delta - \partial_t)u = 0 \\ \text{Waves}: &\quad (\Delta - \partial_t^2)u = 0 \\ \text{Quantum particles}: &\quad \left(\hbar^2 \Delta - \frac{\hbar}{i}\partial_t\right) u = 0 \end{aligned}$$

A key feature of the Laplacian is that it commutes with isometries:

$$\Delta(u \circ T) = \Delta u \circ T, \qquad \Delta(u \circ R) = \Delta u \circ R,$$

or

$$\Delta T^* = T^* \Delta, \qquad \Delta R^* = R^* \Delta$$

where T is translation and R is rotation. In fact, if an operator P commutes with translations T and rotations R, then

$$P = \sum_{j=1}^{n} a_j \Delta^j.$$

Since these are desirable properties for many differential operators in PDE, the Laplacian arises very naturally. We will be interested in the eigenvalue problem

$$-\Delta u_k = \lambda_k^2 u_k.$$

It turns out that there will exist an orthonormal basis $\{u_k\}$ of $L^2(\Omega)$, and we get solutions to our previous equations of the form

$$u(x,t) = \sum_k a_k \alpha_k(t) u_k(x),$$

where $a_k = \langle u(\cdot,0), u_k \rangle$, and

$$\alpha_k(t) = \begin{cases} e^{-\lambda_k^2 t} & \text{heat} \\ e^{i\lambda_k t} & \text{wave} \\ e^{\frac{i}{\hbar}\lambda_k^2 t} & \text{Schrödinger} \end{cases}$$

We will study the following topics:

1. Manifolds and Riemannian geometry
2. General properties of eigenfunctions
3. Spectral theory
4. Functions of Δ (propagators)
5. Weyl's law
6. Ergodicity of eigenfunctions (week 2)
7. Sup norms of eigenfunctions (week 2)

First, we recall the definition of a smooth manifold.

Definition 1. *A smooth n-dimensional manifold M is a Hausdorff, second-countable topological space endowed with a set of charts $\{(U_\alpha, \varphi_\alpha)\}$ of open sets U_α in M and homeomorphisms $\varphi_\alpha : U_\alpha \to \varphi(U_\alpha) \subset \mathbb{R}^n$ such that M is equipped with a smooth structure, i.e.*

1. $M = \bigcup_\alpha U_\alpha$
2. *If $U_\alpha \cap U_\beta \neq \emptyset$, then*

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

   is smooth, with smooth inverse. We say that the two charts are compatible.

3. *If (V, ψ) is a chart compatible with every $(U_\alpha, \varphi_\alpha)$, then $(V, \psi) \in \{(U_\alpha, \varphi_\alpha)\}$.*

Example: Some common examples of manifolds:

1. $M = \mathbb{R}^n$. We can take the single chart $(\mathbb{R}^n, \mathrm{id})$
2. $M = S^1$. We need two charts, which we define via their inverse maps. The first is φ_1 defined as $\varphi_1^{-1}(\theta) = (\sin\theta, \cos\theta)$, with $\theta \in (-\pi, \pi)$. This misses a single point. We take another chart φ_2 defined as $\varphi_2^{-1}(\theta) = (\sin\theta, \cos\theta)$, with $\theta \in (0, 2\pi)$. We cannot include the endpoints in the θ intervals, so we will not have open sets. If we fatten the sets, we lose injectivity. With that being said, it will suffice to integrate on a single chart since each chart misses a set of measure zero.

3. $M = S^2$. An example of a chart is given by $\varphi^{-1}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where θ ranges from 0 to π and ϕ from 0 to 2π. This covers half of the sphere, missing a great circle. In total, we will need 6 charts, although it will suffice to integrate on two.

We say that a function $f : M \to \mathbb{R}$ is differentiable at x if there exists a chart (U, φ) such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is differentiable at $\varphi(x)$. If $f : M \to N$, then we say that same if there exists charts (U, φ) of x and (V, ψ) of $f(x)$ so that such that

$$\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \to \mathbb{R}^n$$

is differentiable at $\varphi(x)$.

Consider $M = \mathbb{R}^n$. If $x \in \mathbb{R}^n$, then we often view the tangent space as the space of velocities with initial point x, which we visualize with arrows. Here, we directly have the identification

$$v \in T_x\mathbb{R}^n \mapsto (v^1, \cdots, v^n).$$

This viewpoint is as amenable to general manifolds. Instead, we view them as operators that differentiate in the direction of v. That is, if $C_x^\infty(\mathbb{R}^n)$ denotes the space of germs of smooth functions at x, then $v : C_x^\infty(\mathbb{R}^n) \to \mathbb{C}$ is given by

$$v(f) = v \cdot \nabla f(x) = \sum_{j=1}^n v^j \frac{\partial f}{\partial x^j}(x),$$

and so we define

$$v = \sum_{j=1}^n v^j \frac{\partial}{\partial x^j}\Big|_x.$$

We have a basis given by

$$\left\{ \frac{\partial}{\partial x^j}\Big|_x : j = 1, \cdots, n \right\}.$$

This is the definition that we adopt for the tangent space of a general manifold (specifically, point derivations of germs of C^∞ functions). Let $x \in M$ and $X_x : C_x^\infty(M) \to \mathbb{R}$. We will say that $X_x \in T_xM$, if for any chart (U, φ) about x, we have that

$$X_x(f) = \sum_{j=1}^n X^j \frac{\partial(f \circ \varphi^{-1})}{\partial x^j}(\varphi(x)),$$

where r^j denote the standard Euclidean coordinates. Implicit in this is the definition

$$\frac{\partial}{\partial x^j}\Big|_x : C_x^\infty \to \mathbb{C}$$

given by

$$\frac{\partial}{\partial x^j}\Big|_x f := \frac{\partial}{\partial r^j}\Big|_{\varphi(x)} (f \circ \varphi^{-1}).$$

If we take coordinates $(x^1, \cdots, x^n)$ near x, then

$$T_xM = \left\{ \frac{\partial}{\partial x_j}\Big|_x : j = 1, \cdots, n \right\}.$$

Clearly, if $x \in M$, then we still have an identification $x \mapsto (x^1, \cdots x^n) \in \mathbb{R}^n$ via φ (specifically, we send x to the coordinates of φ in $\mathbb{R}^n$ evaluated at x, i.e. $x^j(x) = r^j \circ \varphi(x)$). If we write $v \in T_xM$ as

$$v = \sum_{j=1}^n v^j \frac{\partial}{\partial x^j}\Big|_x,$$

then we still have the identification $v \mapsto (v^1, \cdots, v^n) \in \mathbb{R}^n$.

The *tangent bundle* of M, denoted TM, is the $2n$-dimensional smooth manifold

$$TM = \coprod_x T_xM = \{(x, v) : x \in M,\ v \in T_xM\}.$$

This comes equipped with a natural projection map $\pi : TM \to M$. We can identify elements of the tangent bundle to $\mathbb{R}^{2n}$ in the way described above.

Next, we define the *cotangent space* as the dual of the tangent space:

$$T_x^*M = (T_xM)^* = \{\xi : T_xM \to \mathbb{R} \text{ linear}\}.$$

We call the dual basis $\{dx^j\big|_x\}$, so that

$$dx^j \left(\frac{\partial}{\partial x^k} \right) = \delta^j_k.$$

In particular, $dx^j(v) = v^j$ for $v \in T_xM$. We have the basis representation

$$\xi = \sum_{j=1}^n \xi_j dx^j \Big|_x,$$

giving an identification $\xi \mapsto (\xi_1, \cdots, \xi_n)$. Note that

$$\xi(v) = \sum_{j=1}^n \xi_j v^j = \langle \xi, v \rangle_{\mathbb{R}^n}.$$

We similarly define the *cotangent bundle*

$$T^*M = \coprod_x T_x^*M = \{(x, \xi) : x \in M,\ \xi \in T_x^*M\}.$$

It comes with a canonical projection map to M, as well, and has a similar coordinate representation to TM.

A *Riemannian manifold* is a pair (M, g) where M is a smooth manifold and g is a *Riemannian metric*. That is, g is a map that associates to each $x \in M$ an inner product (non-degenerate, symmetric, positive-definite, bilinear form)

$$x \mapsto \langle \cdot, \cdot \rangle_{g(x)} : T_xM \times T_xM \to \mathbb{R}$$

such that for any vector fields V, W on M,

$$x \mapsto \langle V(x), W(x) \rangle_{g(x)}$$

is smooth, as a map from M to $\mathbb{R}$. In a coordinate chart, we can write $g(x) = (g_{ij}(x))$ as an $n \times n$ matrix, where

$$g_{ij}(x) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_{g(x)}.$$

The inverse is denoted g^{ij}.

Example: Some common examples of Riemannian manifolds:

1. $M = \mathbb{R}^n$, $g_{\mathbb{R}^n}(x_1, \cdots, x_n) = I_{n \times n}$
2. $M = S^1$, $g_{S^1}(\theta) = 1$.
3. $M = S^2$,
$$g_{S^2}(\theta, \phi) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}.$$

We can use the metric to identify $\xi \in T_x^* M$ with $X \in T_x M$ via the metric (very Riesz-representation-esque):

$$\xi(Y) = g(Y, X).$$

Hence, $\xi_i = \sum_{j=1}^n g_{ij} x^j$. In particular, we can see that

$$g_x(X, X) = \sum_{i,j=1}^n g_{ij}(x) X^i X^j = \sum_{i,j=1}^n g^{ij}(x) \xi_i \xi_j = |\xi|_{g(x)}.$$

Remark: We sometimes refer to this as lowering an index, via the map $\flat : TM \to T^*M$ given by $X^\flat(Y) = g_x(Y, X)$. In coordinates, we have $X^\flat = g(\cdot, X^i \partial_i) = g_{ij} X^i dx^j$. We often write $X^\flat = X_j dx^j$, where $X_j := g_{ij} X^i$ and say that $X^\flat$ is obtained from X by lowering an index. The matrix of $\flat$ in a coordinate basis coincides with g, and since this is invertible, we have an inverse $\sharp$. In coordinates, this is the map $\omega \mapsto \omega^\sharp$ given by $\omega^i := g^{ij} \omega_j$. This is obtained by raising an index. For example, given a (1,2)-type tensor B with components $B_i{}^j{}_k$, we can lower the middle index to get a covariant 3-tensor $B^\sharp$ with components $B_{ijk} = g_{jl} B_i{}^l{}_k$. This is also used to define the gradient on a manifold.

3 Integration and a Local Expression for the Laplacian

From now on, we will assume that M is a compact, oriented Riemannian manifold. Let (U, φ) be a chart, and $f : M \to \mathbb{C}$. Let dv_g denote the *Riemannian volume form*, which has the local coordinate expression $dv_g = \sqrt{\det g(x)} dx^1 \wedge \cdots \wedge dx^n$. Then, the integral of f on $U \subset M$ is defined as

$$\int_U f\, dv_g := \int_{\varphi(U)} (\varphi^{-1})^* f dv_g = \int_{\varphi(U)} f \circ \varphi^{-1}(x^1, \cdots, x^n) \sqrt{\det g(\varphi^{-1}(x))}\, dx^1 \wedge \cdots \wedge dx^n,$$

where we have used the notation x^j for the standard coordinates (really, it should be $x^j(x)$). Next, let $\{(U_\alpha, \varphi_\alpha)\}$ be our charts on M.

Theorem 3.1. *There exists a partition of unity subordinate to the cover given by the charts. That is, there exist smooth, compactly-supported function $\chi_\alpha : M \to [0,1]$ such that* $\operatorname{supp} \chi_\alpha \subset U_\alpha$*, each $x \in M$ has a neighborhood which intersects* $\operatorname{supp} \chi_\alpha$ *for only finitely many α, and*

$$\sum_\alpha \chi_\alpha(x) = 1$$

for all $x \in M$.

Now, we define the integral of f over all of M:

$$\int_M f\, dv_g = \sum_\alpha \int_{U_\alpha} \chi_\alpha f dv_g = \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^* \chi_\alpha f dv_g.$$

We define

$$L^2(M) = \{f : M \to \mathbb{C} | \int_M |f|^2\, dv_g < \infty\},$$

which is equipped with the inner product

$$\langle f, g \rangle_{L^2} = \int_M f \bar{g}\, dv_g.$$

Now, we introduce the Laplacian on a Riemannian manifold. In local coordinates, the Laplacian has the expression

$$\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial}{\partial x^j} \right).$$

This comes from $\Delta_g = \operatorname{div}_g \circ \nabla_g$, which both have local coordinate expressions.

Example: Some common examples:

1. $$\Delta_{\mathbb{R}^n} = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$$

2. $$\Delta_{S^1} = \frac{\partial^2}{\partial \theta^2}$$

3. $$\Delta_{S^2} = \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right)$$

4 Smoothness of Eigenfunctions and Resolvents

Let's recall a few definitions.

$$\begin{aligned} H^2(M) &= \{u \in L^2 : V_1 V_2 u \in L^2(M)\ \forall \text{ vector fields } V_1, V_2\} \\ &= \{u \in L^2 : Op_h(\langle \xi \rangle^2) u \in L^2(M)\} \quad (= H_h^2(M)) \\ \mathcal{D}'(M) &= \{u : C^\infty(M) \to \mathbb{C} |\ \forall \text{ charts } (U, \varphi)\ \ \forall \chi \in C_c^\infty(\varphi(U))\ \ \mathcal{S}(\mathbb{R}^n) \ni \psi \mapsto u(\varphi^*(\chi\psi)) \in \mathcal{S}'(\mathbb{R}^n)\} \\ H_h^s(M) &= \{u \in \mathcal{D}'(M) : Op_h(\langle \xi \rangle^s) u \in L^2(M)\} \text{ with norm } \|u\|_{H_h^s(M)} = \|Op_h(\langle \xi \rangle^s) u\|_{L^2(M)}, \end{aligned}$$

for any $s \in \mathbb{R}$. Note that

$$\langle \xi \rangle^2 = 1 + |\xi|_{g(x)}^2 = 1 + \sum_{i,j=1}^n g^{ij}(x) \xi_i \xi_j.$$

Here, we are using the Weyl quantization. Another way to define distributions is simply as the continuous dual of $C^\infty(M)$.

We will use the semiclassical Laplacian

$$-h^2 \Delta = h \sum_{i,j=1}^n g^{ij} D_i h D_j + \frac{h}{\sqrt{\det g}} \sum_{i,j=1}^n D_i \left(g^{ij} \sqrt{\det g} \right) h D_j.$$

This defines an element of $\Psi_h^2(M)$, and it has principal symbol $|\xi|_{g(x)}^2$. Some facts about this operator:

1. $$-h^2 \Delta_g : L^2(M) \to L^2(M)$$
   is an unbounded operator, with dense domain $C^\infty(M)$.

2. $-h^2 \Delta_g$ is symmetric/formally self-adjoint:
   $$\left\langle -h^2 \Delta u, v \right\rangle_{L^2} = \left\langle u, -h^2 \Delta_g v \right\rangle_{L^2}$$
   for all $u, v \in C^\infty(M)$.

3. $-h^2 \Delta_g$ is not self-adjoint (the domain of the adjoint is H_h^2). It does admit a self-adjoint extension. In fact, such an extension is unique (i.e. $-h^2 \Delta_g$ is essentially self-adjoint).

4. $$-h^2 \Delta_g : L^2(M) \to \mathcal{D}'(M)$$

5. $$-h^2 \Delta_g : H_h^2(M) \to L^2(M)$$
   is bounded

Theorem 4.1 (Smoothness of Eigenfunctions)**.** *Let $z \in \mathbb{C}$ and $u \in L^2(M)$ be such that*

$$(-h^2 \Delta_g - z) u = 0,$$

interpreted in the sense of distributions. Then, $u \in C^\infty(M)$.

Corollary 4.2. *If z is an eigenvalue, then the corresponding eigenfunction is smooth.*

(Proof of theorem).

Goal: Build $Q_N \in \Psi_h^{-N}(M)$ and $R_{N+1} \in \Psi_h^{-(N+1)}(M)$ so that

$$Q_N(-h^2\Delta_g - z) = I - R_{N+1}.$$

That is, we are going to construct a parametrix. If this is true, then $(I - R_{N+1})u = 0 \implies u = R_{N+1}u$. Since $R_{N+1} : L^2 \to H^{N+1}$, Sobolev embedding will imply that $u \in C^\infty$.

Consider the equation

$$Q_0(-h^2\Delta - z) = I - R_1.$$

One might want to try $q_0(x,\xi) = \frac{1}{|\xi|^2 - z}$, but this explodes, so it's no good. Instead, try

$$q_0(x,\xi) = \frac{1}{|\xi|^2 - z}\left(1 - \chi(|\xi|)\right),$$

where $\chi : C_c^\infty(\mathbb{R}) \to [0,1]$ has the property that $\chi \equiv 1$ for $|t| \le T$, where $T > 0$ and $T^2 > z$ is chosen sufficiently large. For large enough T, $q_0 \in S^{-2}(T^*M)$, so we can quantize:

$$Q_0 = Op_h(q_0) \in \Psi_h^{-2}(M).$$

Note that

$$Q_0 Op_h(|\xi|^2 - z) = Op_h(q_0 \# (|\xi|^2 - z)) = Op_h(1 - \chi(|\xi|)) + hB_1 = I + hB_1,$$

where $B_1 \in \Psi_h^{-1}(M)$. Thus,

$$Q_0(-h^2\Delta_g - z) = I - R_1,$$

where $R_1 \in h\Psi_h^{-1}(M)$. If we define $Q_N := \sum_{k=0}^{N} R_1^k Q_0$, then

$$Q_N(-h^2\Delta_g - z) = \sum_{k=0}^{N} R_1^k Q_0(-h^2\Delta - z) = I - R_1^{N+1} \implies R_{N+1} = R_1^{N+1}.$$

□

Remark: Note that

$$\|R_N\|_{L^2 \to L^2} = \mathcal{O}\left(\frac{h^N}{|\operatorname{Im} z|^{L_N}}\right)$$

for some L_N.

Next, we get a resolvent estimate.

Theorem 4.3. *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then, $-h^2\Delta_g - z : H_h^2(M) \to L^2(M)$ is invertible, and we have the resolvent estimate*

$$\left\|(-h^2\Delta_g - z)^{-1}\right\|_{L^2 \to H_h^2} = \mathcal{O}\left(\frac{1}{|\operatorname{Im} z|} + 1\right).$$

Proof. First, we show that the operator is injective. Let $u \in \ker(-h^2\Delta_g - z)$. It follows from Theorem 4.1 that $u \in C^\infty$, and using symmetry,

$$0 = \left\langle(-h^2\Delta_g - z)u, u\right\rangle = \left\langle u, (-h^2\Delta_g - \bar{z})u\right\rangle = \langle u, (z-\bar{z})u\rangle = -2i \operatorname{Im} z \left\|u\right\|_{L^2}^2.$$

Hence, $u \equiv 0$. Now, we check surjectivity. Call $K = (-h^2\Delta - z)(C^\infty(M))$, and so $L^2 = \bar{K} \oplus K^\perp$. To that end, suppose that $u \in K^\perp$. Then, $\langle u, (-h^2\Delta_g - z)v\rangle = 0$ for all $v \in C^\infty(M)$ (that is, $(-h^2\Delta_g - \bar{z})u = 0$ in $\mathcal{D}'$). Hence, $u \in C^\infty$, and repeating step one yields that $u \equiv 0$. In particular, we get that $\bar{K} = L^2$. Since $C^\infty \subset H_h^2$ is dense and $-h^2\Delta_g - z : H_h^2 \to L^2$ is bounded and injective, it follows that $L^2 = \bar{K} = (-h^2\Delta_g - z)(H_h^2)$. Thus, $-h^2\Delta_g - z : H_h^2 \to L^2$ is bijective, which means that the resolvent $(-h^2\Delta_g - z)^{-1} : L^2 \to H_h^2$ exists.

To see the boundedness, first check that if $u \in C^\infty(M)$, then

$$\left\|(-h^2\Delta_g - z)u\right\|_{L^2} = \left\|(-h^2\Delta - \operatorname{Re} z)u\right\|_{L^2} + \left\|(\operatorname{Im} z)u\right\|_{L^2} \geq |\operatorname{Im} z| \left\|u\right\|_{L^2}.$$

If we take $v \in K$, then we will get that

$$\left\|v\right\|_{L^2} \geq |\operatorname{Im} z| \left\|(-h^2\Delta_g - z)^{-1}v\right\|_{L^2}.$$

By density of $C^\infty(M)$ in $H_h^2(M)$ and the fact that $\Delta\big|_{H_h^2}$ is bounded, this is true for any $v \in H_h^2$.

Going back to earlier notation, it is enough to show that

$$\left\|u\right\|_{H_h^2} \leq C\left(\left\|(-h^2\Delta_g - z)u\right\|_{L^2} + \left\|u\right\|_{L^2}\right)$$

for all $u \in H_h^2$. One could directly apply the elliptic estimate for this. Alternatively,

$$\begin{aligned}\left\|u\right\|_{H_h^2} &= \left\|Op_h(\langle\xi\rangle^2)u\right\|_{L^2} \\ &\leq \left\|Op_h(\langle\xi\rangle^2)Q_1(-h^2\Delta_g - z)u\right\|_{L^2} + \left\|Op_h(\langle\xi\rangle^2)R_2u\right\|_{L^2} \\ &\leq C\left(\left\|(-h^2\Delta_g - z)u\right\|_{L^2} + \left\|u\right\|_{L^2}\right),\end{aligned}$$

where we have used that

$$Op_h(\langle\xi\rangle^2)Q_1, Op_h(\langle\xi\rangle^2)R_2 \in \Psi_h^0$$

and applied the Calderón-Vaillancourt theorem. All together, we have shown that

$$|\operatorname{Im} z|\left(C^{-1}\left\|(-h^2\Delta_g - z)^{-1}v\right\|_{H_h^2} - \left\|v\right\|_{L^2}\right) \leq \left\|v\right\|_{L^2},$$

or

$$\left\|(-h^2\Delta_g - z)^{-1}v\right\|_{H_h^2} \leq C\left(\frac{1}{|\operatorname{Im} z|} + 1\right)\left\|v\right\|_{L^2}$$

□

5 Spectral Theory and Functional Calculus

Theorem 5.1 (Spectral Theorem).

1. *$-h^2\Delta_g$ has a* ***unique*** *self-adjoint extension from $C^\infty(M)$ with domain $H_h^2(M)$.*
2. *For fixed h, there exists an orthonormal basis $\{u_k(h)\}$ of $L^2(M)$, and $\{E_k(h)\} \subset [0,\infty)$, with $E_k(h) \to \infty$ as $k \to \infty$, such that*

$$-h^2\Delta_g u_k(h) = E_k(h) u_k(h).$$

Furthermore, $u_k(h) \in C^\infty$ and $\{E_k(h)\} = \operatorname{spec}(-h^2\Delta_g)$.

Proof.

1. Sketch: Due to the fact that

$$(-h^2\Delta_g)^* \pm i)u = 0 \implies u = 0$$

for $u \in H_h^2$, a result from functional analysis implies that it has a unique self-adjoint extension. The specific functional analytic result says that for a symmetric operator T,

$$T \text{ closed and } \ker(T^* \pm i) = \{0\} \iff T \text{ is self-adjoint.}$$

2. Note that while $-h^2\Delta_g - z$ is not self-adjoint, it is normal. This implies that the resolvent is normal, as well. We claim that $(-h^2\Delta_g - z)^{-1}$ is a compact operator. We can use Rellich's theorem, but let's not do that. Since

$$(-h^2\Delta_g - z)^{-1} = Q_0 + R_1(-h^2\Delta_g - z)^{-1} = \text{compact} + \text{compact} \circ \text{bounded},$$

it follows that the above is compact. Since $(-h^2\Delta_g - z)^{-1}$ is compact and normal, the spectral theorem implies that there exists an orthonormal basis $\{u_k\}$ of L^2 and a discrete set of eigenvalues $\{\mu_k\}$ such that $\mu_k \to 0$ and

$$(-h^2\Delta_g - z)^{-1} u_k = \mu_k u_k$$

(all h-dependent). Clearly, they must all be non-zero, or otherwise the resolvent would fail to exist. These eigenvalues constitute the entirety of the spectrum of the resolvent by the spectral theorem.

If we take $z = i$, then we have that

$$-h^2\Delta_g u_k = \left(\frac{1}{\mu_k} + i\right) u_k.$$

Here, $E_k = \frac{1}{\mu_k} + i \to \infty$ as $k \to \infty$. The fact that they are real and non-negative follows from the fact that $-h^2\Delta_g$ is positive-definite.

□

We also recall that these eigenfunctions are smooth must be smooth. Next, we note the following basis representation for the Laplacian.

Corollary 5.2.

$$-h^2\Delta_g = \sum_{k=1}^{\infty} E_k(h) u_k(h) \otimes \overline{u_k(h)},$$

where we use the notation

$$u \otimes v(\varphi) = u \int_M v\varphi \, dv_g.$$

In particular, if $v = \sum_k \langle v, u_k \rangle u_k$, *then*

$$-h^2\Delta_g v = \sum_{k=1}^{\infty} E_k(h) \langle v, u_k(h) \rangle u_k(h).$$

Let $f \in L^\infty(\mathbb{R})$. We can define a bounded operator $f(-h^2\Delta_g) : L^2 \to L^2$ by

$$f(-h^2\Delta_g) = \sum_{k=1}^{\infty} f(E_k(h)) u_k(h) \otimes \overline{u_k(h)}.$$

For example, the function $f(t) = \frac{1}{t-z}$ generates the resolvent. Recall the concept of an *almost-analytic extension.*

Proposition 5.3. *Let* $f \in \mathcal{S}(\mathbb{R}^n)$, *and fix a cutoff* $\chi \in C_c^\infty((-1,1))$ *such that* $\chi \equiv 1$ *on* $[-1/2, 1/2]$. *Then,*

$$\tilde{f}(z) := \frac{1}{2\pi}\chi(y) \int_{\mathbb{R}} \chi(y\xi)\hat{f}(\xi) e^{i\xi(x+iy)} \, d\xi$$

is an almost-analytic extension of f *to the complex plane. That is,* $\tilde{f} \in C^\infty(\mathbb{C})$, $\tilde{f}\big|_{\mathbb{R}} = f$, $\operatorname{supp} \tilde{f} \subset \{z \in \mathbb{C} : |\operatorname{Im} z| \le 1\}$, *and* $\bar{\partial}_z \tilde{f}(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$, *where* $\bar{\partial}_z = \frac{1}{2}(\partial_x + i\partial_y)$ *is the Cauchy-Riemann operator.*

The proof is straightforward (see Theorem 3.6 in Zworski). Smoothness is obvious, as is the support property (due to the support of χ). The restriction property follows from the Fourier inversion formula. To see the last part, just apply $\bar{\partial}_z$ and integrate by parts. This gives us a nice formula for our functional calculus.

Theorem 5.4 (Helffer-Sjöstrand Formula)**.** *Let* $f \in \mathcal{S}(\mathbb{R})$. *Then,*

$$f(-h^2\Delta_g) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z) (-h^2\Delta_g - z)^{-1} \, dm,$$

where $dm = dxdy$.

I believe that the *idea* for why this formula makes sense (i.e. is well-defined) is that one can pair the resolvent estimates with the properties of the almost-analytic extension. As pointed out by Jared Wunsch, it is a bit unclear that this formula *does* make sense for $f \in \mathcal{S}$, as it appears as if we need to decay when we perturb slightly off of the real axis and go to infinity horizontally (where our resolvent estimates degenerate). It is likely that one can modify the almost-analytic extension to do this. One can certainly do this if $f \in C_c^\infty$ (and this is how the theorem was originally stated). Zworski does state it for Schwartz functions, though. I believe that it's true in view of the more general results in the paper "The Functional Calculus" by Davies (also, it seems to follow from the multiplication version of the spectral theorem for self-adjoint operators).

Proof. First, we claim that, for all $t \in \mathbb{R}$,

$$f(t) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\bar{\partial}_z \tilde{f}(z)}{t - z} \, dm.$$

If this holds, then using how we originally defined $f(-h^2\Delta_g)$,

$$\begin{aligned} f(-h^2\Delta_g) &= \sum_{k=1}^{\infty} \left(\frac{1}{\pi} \int_{\mathbb{C}} \frac{\bar{\partial}_z \tilde{f}(z)}{E_k(h) - z} \, dm \right) u_k(h) \otimes \overline{u_k(h)} \\ &= \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z) \left[\sum_{k=1}^{\infty} \frac{u_k(h) \otimes \overline{u_k(h)}}{E_k(h) - z} \right] dm \\ &= \frac{1}{\pi} \int_{\mathbb{C}} \partial_z \tilde{f}(z) (-h^2\Delta_g - z)^{-1} \, dm. \end{aligned}$$

We were able to swap the sum and integral using the properties of $\tilde{f}$.

To prove the claim, the fact that $\bar{\partial}_z (t - z)^{-1} = 0$ and Green's theorem to obtain that

$$\begin{aligned} \frac{1}{\pi} \int_{\mathbb{C}} \frac{\bar{\partial}_z \tilde{f}(z)}{t - z} \, dm &= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{C} \setminus B(t,\varepsilon)} \frac{\bar{\partial}_z \tilde{f}(z)}{t - z} \, dm = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{C} \setminus B(t,\varepsilon)} \bar{\partial}_z \left(\frac{\tilde{f}(z)}{t - z} \right) dm \\ &= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \oint_{\partial B(t,\varepsilon)} \frac{\tilde{f}(z)}{t - z} \, dz = - \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_0^{2\pi} \frac{\tilde{f}(t + \varepsilon e^{-i\theta})}{t - t - \varepsilon e^{-i\theta}} i \varepsilon e^{-i\theta} \, d\theta \\ &= f(t) \end{aligned}$$

Justifying the last equality comes the expansion $\tilde{f}(z) = f(t) + \mathcal{O}(\varepsilon)$ on $\partial B(t, \varepsilon)$ via the given change of variables and Taylor's theorem. □

This is a very useful formula, and it allows us to prove that our operator is pseudodifferential.

Theorem 5.5. *If $f \in \mathcal{S}(\mathbb{R})$, then*

$$f(-h^2\Delta_g) \in \Psi_h^{-\infty}(M),$$

and it has principal symbol

$$\sigma(f(-h^2\Delta_g)) = f(|\xi|^2_{g(x)}).$$

The first part is the hard part. We will split the proof into the next two lectures.

Proof. We will do the second part first, assuming the first to be true for now. Using our new formula, we see that

$$\begin{aligned} f(-h^2\Delta_g) &= \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)(-h^2\Delta_g - z)^{-1}\, dm = \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)\left[Q_0(z) + (-h^2\Delta_g - z)^{-1}R_1(z)\right] dm \\ &= \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)\left[Op_h\left(\frac{1}{|\xi|^2 - z}\right) + (-h^2\Delta_g - z)^{-1}R_1(z)\right] dm \\ &= Op_h\left(\frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar{\partial}_z \tilde{f}(z)}{|\xi|^2 - z}\right) dm + \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)(-h^2\Delta_g - z)^{-1}R_1(z)\, dm \\ &= Op(f(|\xi|^2)) + \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)(-h^2\Delta_g - z)^{-1}R_1(z)\, dm. \end{aligned}$$

If we can show that the integral piece is $\mathcal{O}_{L^2\to L^2}(h)$, then we will have that $f(|\xi|^2)$ is the principal symbol. Indeed, it follows from the resolvent being bounded by $1 + |\operatorname{Im} z|^{-1}$ on L^2 and R_1 by $h|\operatorname{Im} z|^{-L_1}$, in conjunction with $\tilde{f}$ being $\mathcal{O}(|\operatorname{Im} z|^{\infty})$.

Alternatively, use the parametrix contruction from before *without* the cut-off (we are considering $z \in \mathbb{C} \setminus \mathbb{R}$, so we don't need to worry about singularities), and compute that

$$\left(Op_h(|\xi|^2 - z)\right)^{-1} = (-h^2\Delta_g - z)^{-1} + \mathcal{O}_{L^2\to L^2}\left(h|\operatorname{Im} z|^{-L-1}\right),$$

some $L > 0$. Now, apply Helffer-Sjöstrand. ☐

We'll pick up here next time.

6 Traces and Weyl's Law

We left off with Theorem 5.5. We haven't proven the first part, yet. We will provide a sketch.

Proof of Remainder of Theorem 5.5. Here are the general steps:

Step 1: Show that if $f(-h^2\Delta_g) \in \Psi_h(M) := \Psi_h^0(M)$, for all $f \in \mathcal{S}(\mathbb{R})$, then $f(-h^2\Delta_g) \in \Psi_h^{-\infty}(M)$.

Fix $k \in \mathbb{N}$, and note that $g(t) = (t-i)^k f(t) \in \mathcal{S}(\mathbb{R})$, so we can define

$$g(-h^2\Delta_g) = (-h^2\Delta_g - i)^k f(-h^2\Delta_g) \in \Psi_h(M).$$

Hence,

$$f(-h^2\Delta_g) = (-h^2\Delta_g - i)^{-k} g(-h^2\Delta_g) : H_h^{-k} \to H_h^k.$$

Thus, $f(-h^2\Delta_g) \in \Psi_h^{-\infty}$.

Step 2: This has two parts. The first is to show that if $\chi_1, \chi_2 \in C^\infty(M)$, and they have disjoint supports, then

$$\chi_1 f(-h^2\Delta_g)\chi_2 = \mathcal{O}_{H^{-N}\to H^N}(h^N)$$

for all $N \in \mathbb{N}$ (note the remainder condition is equivalent to the condition that $\chi_1 f(-h^2\Delta_g)\chi_2 \in h^\infty \Psi_h^{-\infty}(M)$). The second is to show that $A := (\varphi^{-1})^* \chi f(-h^2\Delta_g)\chi\varphi^* \in \Psi_h(\mathbb{R}^n)$, for all cut-off charts (φ, χ). By definition, these conditions and a standard partition of unity argument imply that $f(-h^2\Delta_g) \in \Psi_h(M)$.

Step two is hard, and we will hand-wave a bit (details in Zworski). For the first part (i.e. the *pseudolocality*), we note that if we look at $\chi_1 f(-h^2\Delta_g)\chi_2$, then by the Helffer-Sjöstrand formula, we can consider the integral kernel $\chi_1(-h^2\Delta_g - z)^{-1}\chi_2$. Using the parametrix, we can consider

$$\chi_1 Q_m(z)\chi_2 + \chi_1(-h^2\Delta_g - z)^{-1} R_{m+1}(z)\chi_2,$$

where $m \in \mathbb{N}$. Due to the disjoint supports of the cutoffs, the first term is

$$\mathcal{O}_{H_h^{-N}\to H_h^N}\left(h^N |\operatorname{Im} z|^{-L_N}\right),$$

and the second is

$$\mathcal{O}_{H_h^{-N}\to H_h^{-N+m+1}}\left(h^{m+1} |\operatorname{Im} z|^{-K_m}\right),$$

for all N. Now, we simply choose m so that $-N+m+1 \geq N$.

For the second part, we will use Beal's theorem, and we will prove part of it here. By the Schwartz kernel theorem, if $A : \mathcal{S} \to \mathcal{S}'$ is continuous and linear, then it has a Schwartz kernel $K_A(x,y) \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$. Take $h = 1$ (one can use semiclassical rescaling to justify this). If

$$a(x,\xi) = \int e^{-i\xi\omega} K_A\left(x + \frac{\omega}{2}, x - \frac{\omega}{2}\right) d\omega \quad \text{(changing variables in Weyl quantization)},$$

then $A = Op_1^w(a)$. We want to show that $a \in S^0(\mathbb{R}^{2n})$. In particular, we will show that

$$\|\operatorname{ad}_{\omega_1} \cdots \operatorname{ad}_{\omega_N} Op_h(a)\|_{L^2\to L^2} \leq C_N h^N,$$

where $\omega_j \in \{x_1, \cdots, x_n, hD_{\xi_1}, \cdots, hD_{\xi_n}\}$ and $\operatorname{ad}_A B = [A,B]$. If we show this, then we can combine it with the facts that

$$Op_1(D_{x_j}a) = -\operatorname{ad}_{D_{\xi_j}} A$$

and

$$Op_1(D_{\xi_j}a) = \operatorname{ad}_{x_j} A$$

to prove that

$$\|Op_1(\partial^\alpha a)\|_{L^2\to L^2} \le C_\alpha$$

for all α (this would only imply that $a \in S(1) = S^0_{0,0}$, but one one can do better to get $a \in S^0 = S^0_{1,0}$).

If $N = 1$, for example, then $\operatorname{ad}_{\omega_1} A$ becomes $\operatorname{ad}_{\widetilde{\omega_1}} \chi f(-h^2\Delta_g)\chi$ when being sent back to the manifold, and by the Helffer-Sjöstrand formula, one can analyze

$$\operatorname{ad}_{\widetilde{\omega_1}}(-h^2\Delta_g - z)^{-1} = -(-h^2\Delta_g - z)^{-1}\operatorname{ad}_{\widetilde{\omega_1}}(-h^2\Delta_g)(-h^2\Delta_g - z)^{-1} = \mathcal{O}_{L^2\to L^2}\left(h|\operatorname{Im} z|^{-2}\right).$$

It now follows from using properties of the almost-analytic extension. □

Let $A : H \to H$ be compact. Then, A^*A is compact and self-adjoint, so the spectral theorem guarantees eigenvalues $\mu_0^2 \ge \mu_1^2 \ge \cdots \to 0$. In this case, we say that A is *trace class* (denoted $A \in \operatorname{Tr}(H)$) if

$$\sum_{j=0}^\infty |\mu_j| < \infty.$$

If this is the case, then we define

$$\|A\|_{\operatorname{Tr}} = \sum_{j=0}^\infty |\mu_j|.$$

This is a norm, and it forms $\operatorname{Tr}(H)$ into a Banach space. We define the *trace* of a trace class operator as

$$\operatorname{Tr}(A) = \sum \langle Ae_j, e_j\rangle,$$

where $\{e_j\}$ is an orthonormal basis of H. One can check that this is independent of choice of basis. Formally,

$$\operatorname{Tr}(f(-h^2\Delta_g)) = \sum f(E_k(h)).$$

It turns out, although we will not prove it, that if $f \in \mathcal{S}(\mathbb{R})$, then $f(-h^2\Delta_g) \in \operatorname{Tr}(L^2(M))$. This comes from the fact that if $a \in \mathcal{S}(\mathbb{R}^{2n})$, then $Op_h^w(a) \in \operatorname{Tr}(L^2(\mathbb{R}^n))$. In this case, one can check that

$$\begin{aligned}\operatorname{Tr}(Op_h(a)) &= \int_{\mathbb{R}^n} K_{Op_h(a)}(x,x)\,dx = (2\pi h)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{i}{h}(x-x)\xi} a\left(\frac{x+x}{2},\xi\right)dxd\xi \\ &= (2\pi h)^{-n}\iint_{\mathbb{R}^{2n}} a(x,\xi)\,dxd\xi.\end{aligned}$$

If we're on a manifold, then $\mathbb{R}^{2n}$ changes to T^*M. In particular,

$$\operatorname{Tr}(f(-h^2\Delta_g)) = (2\pi h)^{-n}\left(\iint_{T^*M} f(|\xi|_g^2)\,dxd\xi + \mathcal{O}(h)\right).$$

Example: If $f = \chi_{[a,b]}$, then

$$\operatorname{Tr}(\chi_{[a,b]}(-h^2\Delta_g)) = \#\{k : a \le E_k(h) \le b\}.$$

There is a problem with this example, namely that f is not Schwartz here. We will want this in the upcoming proof, and the idea is to regularize.

Theorem 6.1 (Weyl's Law)**.**

1. *Suppose that*

$$-h^2\Delta_g u_k(h) = E_k(h)u_k(h).$$

Then,

$$\#\{k : a \le E_k(h) \le b\} = (2\pi h)^{-n} \iint\limits_{T^*M} \chi_{a\le|\xi|_g^2\le b}\, dx d\xi + \mathcal{O}(h^{-n+1}).$$

2. *Suppose that*

$$-\Delta_g u_k = \lambda_k^2 u_k,$$

with $\lambda_k \nearrow \infty$*. Then,*

$$\#\{k : \lambda_k \le \lambda\} = \frac{\lambda^n}{(2\pi)^n} \operatorname{vol}(B_{\mathbb{R}^n}(0,1)) \operatorname{vol}(M) + \mathcal{O}(\lambda^{n-1}).$$

Proof. First, note that (1) $\implies$ (2). Indeed, just take $a = 0, b = 1, h = \frac{1}{\lambda}$. Note that $\lambda_k^2 = \lambda^2 E_k\left(\frac{1}{\lambda}\right)$, or $E_k\left(\frac{1}{\lambda}\right) = \frac{\lambda_k^2}{\lambda^2}$. So, we only need to prove (1). Choose $f_\varepsilon, g_\varepsilon \in \mathcal{S}(\mathbb{R})$ such that

$$f_\varepsilon \le \chi_{[a,b]} \le g_\varepsilon,$$

and $f_\varepsilon, g_\varepsilon \to \chi_{[a,b]}$ pointwise. For example, take g_ε to be smooth and compactly-supported equal to 1 on $[a-\varepsilon, b+\varepsilon]$, and f_ε similar, but equal to 1 on $[a+\varepsilon\frac{a+b}{2}, b-\varepsilon\frac{a+b}{2}]$. Since $f_\varepsilon(-h^2\Delta_g), g_\varepsilon(-h^2\Delta_g) \in \operatorname{Tr}(L^2)$, it follows that

$$\operatorname{Tr}(f_\varepsilon(-h^2\Delta_g)) \le \#\{k : a \le E_k(h) \le b\} \le \operatorname{Tr}(g_\varepsilon(-h^2\Delta_g)).$$

Since $g_\varepsilon \in \mathcal{S}(\mathbb{R})$, $g_\varepsilon(-h^2\Delta_g) \in \Psi_h^{-\infty}$, and it has principal symbol $\sigma(g_\varepsilon(-h^2\Delta_g)) = g_\varepsilon(|\xi|_g^2)$. Taking the trace,

$$\operatorname{Tr}(g_\varepsilon(-h^2\Delta_g)) = (2\pi h)^{-n}\left(\iint\limits_{T^*M} g_\varepsilon(|\xi|_g^2)\, dx d\xi + \mathcal{O}_\varepsilon(h)\right).$$

Finally, note that

$$\operatorname{Tr}(g_\varepsilon(-h^2\Delta_g)) = (2\pi h)^{-n}\left(\iint\limits_{T^*M} \chi_{[a,b]}(|\xi|_g^2)\, dx d\xi + \mathcal{O}(\varepsilon) + \mathcal{O}_\varepsilon(h)\right).$$

Take the limit as $\varepsilon, h \to 0$, then do the same process for f_ε. □