My Analysis Comprehensive Exam Solutions

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Contents

1 How to Read

This document contains my "solutions" to 14 UNC graduate analysis comprehensive exams, as well as 49 additional problems (mostly comp problems). There is a section of solved extra problems (38 problems) and a section of extra problems with hints (11 problems), which I have tried my best to label according to where I got them (from an old in-class test, from an old comp, etc.). This amounts to solutions to $\sim$ 19 comps and $\sim$ 1.5 comps in problems with hints. I have solutions to other problems which are untyped...somewhere. If there is a problem that you're curious about which is not present here, feel free to shoot me an email (if it is older, there is a very good chance that I did it, and if it's newer, I probably looked at it and thought about it a bit). I have also included how I studied for the exam, a categorized list of all of the problems from Spring 2008-Spring 2021, and a brief analysis of some relevant types of problems and tools used to attack them.

The detail of these solutions varies, depending on whether these were written when I ran the comp review sessions or not, whether or not similar problems had been done, etc. For example, I often didn't write out full solutions to contour integrals and conformal map problems if I'd solved a similar problem in the document (I skipped more of the contour integrals: for fully worked out contour integrals, see #8 in Section 4, #6 in Section 11, #6 in Section 13, #5 in Section 17; #6 in Section 5 is mostly worked out, and #7 in Section 10 shows a trick for calculating residues). If I didn't work it out, I hyperlinked it to a relevant problem that I did work out. I also want to point out that some of these solutions (especially those given during the comp review sessions) were written a bit more conversationally, since I was trying to write to the people reading them. I also occasionally demonstrate alternative pathings for portions of problems during the proofs themselves. This obviously is not what I would do for an actual comp, so be sure to make this distinction when such instances occur. Finally, one should note that *some* of the solutions are likely more detail-oriented than could reasonably be expected on a comp. I do not know this for sure, though, since I do not grade them.

Any mistakes are obviously mine. This document is quite large, so mistakes are bound to exist. The exams that I assigned for comp reviews should have better solutions and are marked accordingly in the format *Semester Year*. The other sections are solutions that I wrote when I was studying for the exam myself back in 2018 and have not been proofread. So, they will likely have more mistakes. **Please let me know if/when you find any mistakes, and I will correct them**. In the exams that I assigned, the solutions also frequently contained tips for taking the exam and motivation for the approaches taken (especially the ones that I assigned during my second time leading). Hence, these should be looked at with more priority. I do not remember offhand the precise order that I would go in, but I know that I like to give Fall 2014 (Section 11) first and Fall 2012/Spring 2014 as the last two (Sections 15 and 12, respectively).

2 How I Studied

In order to study for the exam, I took the following steps.

1. I did nearly every problem from January 2018 to some time in the early 2000's. On

the problems that I did not do (often older problems), I made sure that I knew the approach and only skipped if they were repetitive. I timed myself on a lot of these exams (especially on the ones typed here) and gunned for completion within 2 hours (you are allotted three hours for the actual test). Then, I finished any uncompleted problems, vetted my answers (if possible), and typed my solutions.

2. I kept a list of a lot of the comps that I did. I labeled each type of problem and my performance on the problem. I went back to problems that I struggled with after a bit of time and continued to do so until I felt very comfortable with them.

3. Using my log, I checked what types of problems were most prevalent and made sure that I was air-tight on those.

4. I re-did all of the exams from the classes that I took. I re-did a lot of my homework problems, too. Since Jason (Metcalfe) assigned a lot of comp problems from a variety of time periods, this was very good practice.

Doing old comps (within reason) are still highly relevant for preparing for the current day ones. Many of the problems on the exams in the past few years are still versions of problems which are present here (see the Spring 2020 exam, for example). Even if the new problems are not *identical*, doing old comps is a great way to get used to doing difficult problems on the relevant material.

3 Problem Breakdown: Spring 2008-Spring 2021

3.1 Real Analysis Problem Categorization

1. Advanced calculus (count=22)

   - S2021 #1,2, F2020 #1, S2019 #2,3, F2018 #1, S2018 #3, S2017 #2, S2016 #2, F2015 #1, F2014 #1, F2013 #1, S2013 #1, F2012 #4, S2012 #1, F2011 #1, F2010 #3,4, F2009 #3,4, F2008 #2,4

2. Topology (count=3)

   - S2017 #1, F2016 #2, F2009 #1

3. Contraction mapping theorem and ODE theory (count=12)

   - F2020 #3, F2019 #3, F2018 #3, S2017 #3, F2016 #1, F2015 #2, S2014 #4, S2012 #3, F2011 #2, F2010 #1, S2010 #4, S2008 #3, F2008 #3

4. Arzelà-Ascoli theorem (count=5)

   - S2020 #1, S2015 #2, S2014 #2, F2013 #2, S2011 #2

5. Weierstrass approximation theorem and Stone-Weierstrass theorem (count=3)

   - S2016 #1, S2013 #2, S2010 #3

6. Multivariable differentiation, Inverse/Implicit function theorem(s) (count=18)
    - S2021 #3, S2020 #3, S2019 #1, F2018 #2, S2018 #1, F2017 #1, S2016 #4, S2015 #1, F2014 #2, S2013 #4, S2012 #2, S2011 #3, F2010 #2, S2009 #1,4, F2009 #2, S2008 #2,4
7. Measurable sets and functions (count=14)
    - S2021 #4, S2020 #2, F2019 #1, F2018 #4, S2018 #4, F2017 #4, S2016 #3, F2015 #3, S2015 #3, S2014 #1, F2013 #3, F2011 #4, S2011 #1, S2008 #1
8. Integration theory (count=25)
    - F2020 #2,4, S2020 #4, F2019 #2, S2019 #4, S2018 #2, F2017 #2,3, S2017 #4, F2016 #4, F2015 #4, S2015 #4, F2014 #3,4, S2014 #3, F2013 #4, S2013 #3, F2012 #3, S2012 #4, F2011 #3, S2011 #4, F2010 #2, S2010 #1, S2009 #2, F2008 #1
9. Miscellaneous (count=5)
    - F2019 #4, F2016 #3, F2012 #1,2, S2009 #3

3.2 Complex Analysis Problem Categorization

1. Properties of analytic functions (count=46)
    - S2021 #7, F2020 #5,7,8, S2020 #6,8, F2019 #7,8 S2019 #5, F2018 #6,7, S2018 #5,6, F2017 #5,7,8, S2017 #7, S2016 #5,6, F2015 #5, S2015 #5,8, F2014 #5,7, S2014 #6,7, F2013 #7, S2013 #5,6,8, F2012 #8, S2012 #6,7, F2011 #6, S2011 #6,7,8, F2010 #5, S2010 #6,7,8, S2009 #8, F2009 #5, F2008 #5,6,7
2. Harmonic function theory (count=5)
    - S2021 #5, S2020 #7, F2019#6, S2016 #8, F2013 #8
3. Laurent series (count=6)
    - S2020 #5, S2017 #6, S2014 #5, F2013 #5, F2012 #7, F2009 #6
4. Integration, residues (count=19)
    - S2021 #8, F2019#5, S2019 #7, F2018 #5, F2017 #6, S2017 #5, F2016 #7, F2015 #6, S2015 #7, F2014 #6, F2013 #6, S2012 #5, F2011 #7, S2011 #5, F2010 #6, S2010 #5, S2009 #6, F2009 #7, S2008 #8
5. Conformal maps (count=15)
    - S2019 #8, F2018 #8, S2018 #7, S2017 #8, F2016 #6,8, F2015 #8, S2015 #6, F2014 #8, F2012 #6, S2012 #8, F2011 #8, F2010 #7, F2009 #8, S2008 #7

6. Rouché's theorem and the argument principle (count=11)
    - S2019 #6, S2018 #8, F2016 #5, S2014 #8, S2013 #7, F2012 #5, F2011 #5, F2010 #8, S2008 #5,6, F2008 #8
7. Normal families (count=3)
    - S2021 #6, S2016 #7, F2015 #7
8. Riemann mapping theorem (count=3)
    - F2020 #6, S2009 #5,7

3.3 Some Notable Patterns and Useful Tools

Popular types of real analysis problems include

1. Fundamental elementary results (Bolzano-Weierstrass, Fundamental theorem of calculus, nested cell theorem, etc.)
2. Contraction mapping, ODEs; useful tools include
    (a) Not a tool, but you should be able to prove the contraction mapping theorem
    (b) Using contraction mapping to prove existence and uniqueness of integral equations.
    (c) Using contraction mapping to prove ODE existence and uniqueness theory
3. Inverse and implicit function theorem problems; useful tools include the inverse and implicit function theorems
4. Fréchet differentiation problems; useful tools include
    (a) The derivative definition
    (b) Implicit/inverse function theorems
5. Exponential problems; useful tools include
    (a) Power series and the Weierstrass M-test
    (b) Proving equalities of exponential expressions by moving everything to one side and showing that such a function is constant and equal to the identity. Justify inversion to move everything over and prove equality.
    (c) Inverse function theorem (special type of previous problem type)
6. Lebesgue measure; useful tools include
    (a) Caratheodory definition of Lebesgue measure
    (b) Properties of Lebesgue measure (general properties of measure, completeness, translation invariance)

(c) Types of measurable sets (e.g. open and closed sets, intervals, Cartesian products, countable sets are null, combinations using properties of measure)

(d) Cantor set, fat Cantor sets

(e) Existence of non-measurable sets (Vitali set)

7. Lebesgue integration; useful tools include

   (a) Approximation by simple functions

   (b) Monotone convergence theorem

   (c) Fatou's lemma

   (d) Dominated convergence theorem

8. Not a problem type, but knowing relevant density results can be an extremely useful tool.

9. Although not as prevalent pattern-wise, I highly recommend knowing how to apply the Arzelà-Ascoli theorem thoroughly. It pops up every now and then, and Jason emphasizes it a lot in his MATH 653 course.

Popular types of complex analysis problems include

1. Finding all analytic/meromorphic functions satisfying a property; useful tools include

   (a) The Schwarz lemma

   (b) Liouville's theorem

   (c) Looking at power series/Laurent series and analyzing coefficients like in the proof of Liouville's theorem (this is related to the Cauchy integral formula)

   (d) Removable singularity theorem (which you should know how to prove)

2. Counting zeros; useful tools include

   (a) Rouché's theorem

   (b) The argument principle

3. Contour integrals; useful tools include

   (a) Cauchy integral formula

   (b) Residue theory

   (c) Argument principle

4. Not a type of problem, but it is important to know all of the different definitions for holomorphic. One should also know how these relate to fundamental results (e.g. Cauchy integral theorem $\to$ Cauchy integral formula $\to$ holmorphic functions are smooth $\to$ holomorphic functions are analytic)

5. Conformal maps; useful tools include
   (a) Automorphisms of the disk
   (b) Automorphisms of the upper half-plane
   (c) Three point theorem
   (d) Facts about linear fractional transformations and how they send circles/extended lines to circles/extended lines (see Michael Taylor's text for all of the possibilities)
   (e) Standard map (Cayley transform) from the upper half-plane to the unit disk
   (f) Exponentials, logs, trig functions, scalings, translations, rotations, inversion

3.4 Helpful Textbooks

Here are some helpful textbooks (non-exhaustive). All of the ones that I personally referenced are colored red (and I have frequently seen problems from these textbooks pop up on comps). The first two listed books in each sections are the most commonly-used textbooks for the graduate courses.

Real:

1. Michael Taylor's MATH 653 Notes (Course Website)
2. *Analysis II* by Tao
3. *Principles of Mathematical Analysis* by Rudin
4. *III Real Analysis: Measure Theory, Integration, and Hilbert Spaces* by Stein and Shakarchi
5. *Elementary Classical Analysis* by Marsden and Hoffman

Complex:

1. Michael Taylor's MATH 656 Notes (Course Website)
2. *Complex Function Theory* by Sarason
3. *II Complex Analysis* by Stein and Shakarchi
4. *Complex Analysis* by Ahlfors
5. *Basic Complex Analysis* by Marsden and Hoffman
6. *Functions of One Complex Variable* by Conway