AN OVERVIEW OF THE SEMICLASSICAL WEYL QUANTIZATION

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Abstract. The goal of this project is provide an introduction to the Weyl quantization and work up to a proof of the Gårding inequality for an appropriate class of symbols. We will begin with a quick review of the Fourier transform, as well as a semiclassical variant. Next, we will introduce the concept of quantization and various quantization formulas for Schwartz class symbols, with a primary focus on the Weyl quantization. This will include a discussion of the asymptotic expansion of the composition for the Weyl quantization, which demonstrates the important fact that the Poisson bracket at the classical level encodes, essentially, the same information as the commutator does at the quantum level. From here, we will extend our symbols to more general classes and describe various properties of the Weyl quantization on such symbols, such as L^2 boundedness and compactness. Finally, we will move onto invertibility of the Weyl quantization for elliptic symbols, then prove the Gårding inequality, in both a strong and weak form. The appendix includes a detailed proof of the Cotlar-Stein lemma.

In order to go through the above without writing an excessive amount, some results will be stated without proof, and others might be sketched. The goal is to provide a reasonable and somewhat expository overview of the Weyl calculus and reach the Gårding inequality, so some intermediate results cannot be prioritized (many are needed for proof). The results that we prove along the way will primarily be those directly related to functional analysis (and the details are tractable in a short paper), or those that use functional analytic techniques.

Contents

1. A Note on References

First off, I would like to say that this project will primarily following Zworski's manuscript ([7]). I have added in expository details of my own, but I will be primarily proceeding as they did. I have filled in details to proofs as needed (for me), as well as cut certain results or technical lemmas that I thought seemed either (a) to detract from the exposition or (b) were unnecessary to fulfill my goals for the project. I also personally referenced many other texts on pseudodifferential operators and microlocal analysis, even if their influence is not apparent in this write-up, such as [2], [3],[4], and [6].

2. The Fourier Transform

We first recall the definition of the Fourier transform as the operator $\mathcal{F}$ such that

$$\mathcal{F}u(\xi) = \hat{u}(\xi) := \int_{\mathbb{R}^n} u(x)e^{-ix\cdot\xi}\, dx,$$

for $u \in \mathcal{S}(\mathbb{R}^n)$, say (we will just write $\mathcal{S}$ henceforth). The Fourier transform enjoys many useful properties, such as

$$D_\xi^\alpha(\mathcal{F}u) = \mathcal{F}((-x)^\alpha u)$$

and

$$\mathcal{F}(D_x^\alpha u) = \xi^\alpha \mathcal{F}u,$$

where $D^\alpha = \frac{1}{i^{|\alpha|}}\partial^\alpha$ and α is a multi-index. A key feature of this operator is that $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a topological isomorphism, with inverse

$$\mathcal{F}^{-1}u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} u(\xi)e^{ix\cdot\xi}\, d\xi.$$

This gives rise to the famed Fourier inversion formula

$$u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{u}(\xi)e^{ix\cdot\xi}\, d\xi$$

Using the Fourier inversion formula and Fubini's theorem, one readily obtains the Plancheral theorem:

$$\|u\|_{L^2}^2 = \frac{1}{(2\pi)^n}\|\hat{u}\|_{L^2}^2 .$$

Different scaling in the definition of $\mathcal{F}$ would make the Fourier transform unitary, but we will not adopt that convention here. The Fourier transform can further be defined on $\mathcal{S}'$ by duality, as well as extended linearly to L^2 (from the space of functions which are continuous, bounded, L^1, and have L^1 Fourier transform).

The Fourier transform is a powerful tool that allows one to readily obtain solutions to linear PDEs with sufficient regularity. Even more so, it can be viewed as a transformation from the position variable x to the momentum variable ξ, which carries relevance in microlocal analysis, where one works in phase space, which is viewed as the cotangent bundle in the Hamiltonian setting.

Since we will be working in the semiclassical framework, we define the *semiclassical Fourier transform* as a re-scaled version of the Fourier transform,

$$\mathcal{F}_h u(\xi) = \int_{\mathbb{R}^n} u(x)e^{-\frac{i}{h}x\cdot\xi}\, dx,$$

for arbitrary $h > 0$, with analogous inverse

$$\mathcal{F}_h^{-1}u(x) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} u(\xi)e^{\frac{i}{h}x\cdot\xi}\, d\xi.$$

The semiclassical Fourier transform possesses all of the same properties as the standard version, with some scalings by h. From this point forward, we will write either $\mathcal{F}_h f$ or $\hat{f}$ to denote the semiclassical Fourier transform of f.

A classic result that can be proved easily is the *semiclassical uncertainty principle*

$$\frac{h}{2}\left\|u\right\|_{L^2}\left\|\mathcal{F}_h u\right\|_{L^2} \leq \left\|x_j u\right\|_{L^2}\left\|\xi_j \mathcal{F}_h u\right\|_{L^2},$$

which states that one cannot arbitrarily simultaneously localize in both position and momentum. The proof follows directly from computing the commutator

$$[x_j, hD_{x_j}]u = ihu$$

and using the Cauchy-Schwarz inequality.

To understand the asymptotics of the Fourier transform, one employs the principles of stationary phase, with the phase function $x \cdot \xi$, which we will not discuss here.

3. Quantization for Schwartz Symbols

Fix $h > 0$, and let $a \in \mathcal{S}(\mathbb{R}^{2n})$. We call a a *symbol*, and we will often write $a = a(x, \xi)$.

Definition 1. The Weyl quantization is the operator $a^w(x, hD)$ acting on $\mathcal{S}$ by

$$a^w(x, hD)u(x) = \frac{1}{(2\pi h)^n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) u(y)\ dyd\xi.$$

Note that the kernel of the operator is $\mathcal{F}_h^{-1}\left(a\left(\frac{x+y}{2}\right), \cdot\right)(x-y)$. The Weyl quantization is an example of a semiclassical pseudodifferential operator. In fact, one can define more general pseudodifferential operators $Op_t(a)$ by

$$Op_t(a)u(x) = \frac{1}{(2\pi h)^n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} a\left(tx + (1-t)y, \xi\right) u(y)\ dyd\xi.$$

From the perspective of quantum mechanics, we can views the symbol as a classic observable and the operator as a quantum observable.

Proposition 3.1. *Let $a \in \mathcal{S}$.*

1. *One can define $a^w(x, hD) : \mathcal{S}' \to \mathcal{S}$, and such a map is continuous.*
2. *If a is real, then $a^w(x, hD)$ is formally self-adjoint.*

Remark 3.2. We will denote distributional action of $u \in \mathcal{S}'$ and $v \in \mathcal{S}$ by $\langle u, v\rangle$ (the opposite of what we did in-class).

Proof. (1) As stated previously,

$$a^w(x, hD)u(x) = \int\limits_{\mathbb{R}^n} K_t(x, y)u(y)\ dy,$$

where

$$K_t(x, y) = \mathcal{F}_h^{-1} a\left(\left(\frac{x+y}{2}\right), \cdot\right)(x-y).$$

Since $a \in \mathcal{S}$, it follows from the above that $K_t \in \mathcal{S}$, and so defining $a^w(x, hD)u(x) = \langle u, K_t(x, \cdot)\rangle$ yields a continuous map from $\mathcal{S}'$ into $\mathcal{S}$.

(2) For $u, v \in \mathcal{S}$,

$$
\begin{aligned}
(a^w(x,hD)u,v)_{L^2} &= \int\limits_{\mathbb{R}^n} \left(\frac{1}{(2\pi h)^n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} a\left(\frac{x+y}{2},\xi\right) u(y)\ dyd\xi \right) \overline{v(x)}dx \\
&= \int\limits_{\mathbb{R}^n} u(y) \left(\frac{1}{(2\pi h)^n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} a\left(\frac{x+y}{2},\xi\right) \overline{v(x)}\ dxd\xi \right) dy \\
&= \int\limits_{\mathbb{R}^n} u(y) \overline{\left(\frac{1}{(2\pi h)^n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \overline{e^{\frac{i}{h}(x-y)\cdot\xi} a\left(\frac{x+y}{2},\xi\right)} v(x)\ dxd\xi \right)} dy \\
&= \int\limits_{\mathbb{R}^n} u(y) \overline{\left(\frac{1}{(2\pi h)^n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} a\left(\frac{x+y}{2},\xi\right) v(x)\ dxd\xi \right)} dy \\
&= (u, a^w(x,hD)v)_{L^2},
\end{aligned}
$$

using Fubini's theorem on the second line and that a is real on the second-to-last.

□

Remark 3.3. Later, we will show, for certain symbols (a fairly broad class, in fact) that $a^w(x,hD)$ is bounded on L^2, in which case a real will imply that $a^w(x,hD)$ is self-adjoint.

In a similar manner as above, one obtains the following proposition.

Proposition 3.4. *If $a \in \mathcal{S}'$, then one can define $a^w(x,hD) : \mathcal{S} \to \mathcal{S}'$, and it is continuous.*

The only difference here is that one interprets K_t is a distributional way, giving $K_t \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$. Then, we define $a^w(x,hD)$ via $(a^w(x,hD)u)(v) = K_t(v \otimes u)$.

We will provide a few explicit examples of the Weyl quantization of certain symbols, some of which we will need later on. First, fix $(x^*, \xi^*) \in \mathbb{R}^{2n}$, and define the linear symbol $l(x,\xi) = x^* \cdot x + \xi^* \cdot \xi$. We will occasionally identify l with the point (x^*, ξ^*), and it should be obvious from context when we are doing so.

Proposition 3.5.

(1) *Suppose that $c = (c_1(x), \cdots, c_n(x))$ is independent of ξ. Then,*

$$\langle c, hD\rangle^w = \frac{h}{2}\sum_{j=1}^n (D_{x_j}c_j + c_j D_{x_j})$$

.

(2)

$$(D_{x_j}a)^w = [D_{x_j}, a^w]$$

and

$$h(D_{\xi_j}a)^w = -[x_j, a^w].$$

(3) *Fix $(x^*, \xi^*) \in \mathbb{R}^{2n}$, and let l be the corresponding linear symbol. Then,*

$$\left(e^{\frac{i}{h}l}\right)^w (x,hD) = e^{\frac{i}{h}l(x,hD)},$$

where
$$e^{\frac{i}{h}l(x,hD)}u(x) = e^{\frac{i}{h}x^*\cdot x+\frac{i}{2h}x^*\cdot\xi^*}u(x+\xi^*).$$

(4) *If $l, m \in \mathbb{R}^{2n}$, then*
$$e^{\frac{i}{h}l(x,hD)}e^{\frac{i}{h}m(x,hD)} = e^{\frac{i}{2h}\sigma(l,m)}e^{\frac{i}{h}(l+m)(x,hD)},$$
where σ is the symplectic product of $\mathbb{R}^{2n}$ (that is, if $z = (x,\xi)$ and $w = (y,\eta)$, then $\sigma(z,w) = \xi\cdot y - x\cdot\eta$).

We will not write out the proofs, as they are essentially just calculations, but we will give the idea:

(1) This follows directly from integration by parts and that if $a(x,\xi) = a(x)$, then $a^w(x,hD) = M_a$, where M_a is the multiplication operator by a (this is, in fact, true for $Op_t(a)$ for any $t \in [0,1]$).
(2) This similarly employs integration by parts. The first identity is also proved using the splitting $\left(D_{x_j}a\left(\frac{x+y}{2},\xi\right)\right) = \left(D_{x_j}+D_{y_j}\right)a\left(\frac{x+y}{2},\xi\right)$
(3) This is proved in a less obvious manner. One considers the PDE
$$\begin{cases} ih\partial_t v + l(x,hD)v = 0 \\ v(0) = u \end{cases},$$
which has the unique solution $v(x,t) = e^{\frac{it}{h}l(x,hD)}u$. Since $l(x,hD) = x^*\cdot x + \xi^* hD$, it follows that
$$v(x,t) = e^{\frac{it}{h}x^*\cdot x+\frac{it^2}{2h}x^*\cdot\xi^*}u(x+t\xi^*).$$
Now, just evaluate at $t=1$. We get that this equals $\left(e^{\frac{i}{h}l}\right)^w$ by straightforward calculation, which uses that, as elements of $\mathcal{S}'$,
$$\delta_x = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi}\, d\xi.$$
(4) This follows from comparing the formulas. Note that, here, we are identifying $l(x,\xi) = x^*\cdot x + \xi^*\cdot\xi$ with (x^*,ξ^*) and similarly for m when defining $\sigma(l,m)$.

Something valuable to observe is that
$$\mathcal{F}_h^{-1}a^w(x,hD)\mathcal{F}_h = a^w(hD,-x).$$
If we write $z = (x,\xi)$, then $a(\xi,-x) = J^*a(x,\xi) =: a(Jz)$, where J^*a denotes the pullback of a under J, and
$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$
is the $2n\times 2n$ matrix such that $\sigma(z,w) = (Jz)\cdot w$. Since J is skew-symmetric, $-J$ is the matrix representation of the symplectic form in the standard basis $\{e_1,\cdots,e_{2n}\}$. So, one can view the Weyl quantization of a Schwartz symbol in the semiclassical Fourier basis as being the Weyl quantization of the pullback of the symbol under the symplectic map.

Before moving on to general symbols, we will discuss composition. We seek to find the symbol $a\#b$ so that $a^w b^w = (a\#b)^w$.

Proposition 3.6. *Define*

$$\hat{a}(l) = \int\limits_{\mathbb{R}^{2n}} a(x,\xi) e^{-\frac{i}{h} l(x,\xi)} \, dx d\xi,$$

for $a \in \mathcal{S}$ and $l \in \mathbb{R}^{2n}$. Then,

$$a^w(x, hD) = \frac{1}{(2\pi h)^{2n}} \int\limits_{\mathbb{R}^{2n}} \hat{a}(l) e^{\frac{i}{h} l(x,hD)} \, dl.$$

Proof. By the Fourier inversion formula,

$$a(x,\xi) = \frac{1}{(2\pi h)^{2n}} \int\limits_{\mathbb{R}^{2n}} \hat{a}(l) e^{\frac{i}{h} l(x,\xi)} \, dl.$$

The result for follows from part (3) of Proposition 2.5. □

Remark 3.7. If $a \in \mathcal{S}'$, then the result holds in the sense of tempered distributions. That is, for $u, v \in \mathcal{S}$, $\langle e^{\frac{i}{h} l(x,hD)} u, v\rangle$ is Schwartz in $l \in \mathbb{R}^{2n}$, and

$$\langle a^w(x,hD)u, v\rangle = \frac{1}{(2\pi h)^{2n}} \langle \hat{a}, \langle e^{\frac{i}{h}(\cdot)(x,hD)} u, v\rangle\rangle.$$

Let $A(D) = \frac{1}{2}\sigma(D_x, D_\xi, D_y, D_\eta)$. Now, we are prepared to define composition.

Proposition 3.8. *If $a, b \in \mathcal{S}$, then*

$$a^w(x,hD) b^w(x,hD) = (a\#b)^w(x,hD),$$

where

$$a\#b(x,\xi) = e^{ihA(D)}(a(x,\xi)b(y,\eta))\big|_{\substack{y=x\\ \eta=\xi}}$$

Proof. By the previous proposition,

$$a^w(x,hD) = \frac{1}{(2\pi h)^{2n}} \int\limits_{\mathbb{R}^{2n}} \hat{a}(l_1) e^{\frac{i}{h} l_1(x,hD)} \, dl_1,$$

$$b^w(x,hD) = \frac{1}{(2\pi h)^{2n}} \int\limits_{\mathbb{R}^{2n}} \hat{b}(l_2) e^{\frac{i}{h} l_2(x,hD)} \, dl_2,$$

and so, using Proposition 3.5, we have

$$\begin{aligned} a^w(x,hD) b^w(x,hD) &= \frac{1}{(2\pi h)^{4n}} \int\limits_{\mathbb{R}^{2n}} \int\limits_{\mathbb{R}^{2n}} \hat{a}(l_1)\hat{b}(l_2) e^{\frac{i}{h} l_1(x,hD)} e^{\frac{i}{h} l_2(x,hD)} \, dl_1 dl_2 \\ &= \frac{1}{(2\pi h)^{4n}} \int\limits_{\mathbb{R}^{2n}} \int\limits_{\mathbb{R}^{2n}} \hat{a}(l_1)\hat{b}(l_2) e^{\frac{i}{2h}\sigma(l_1,l_2)} e^{\frac{i}{h}(l_1+l_2)(x,hD)} \, dl_1 dl_2 \\ &= \frac{1}{(2\pi h)^{2n}} \int\limits_{\mathbb{R}^{2n}} \hat{K}(r) e^{\frac{i}{h} r(x,hD)} \, dr, \end{aligned}$$

where

$$\hat{K}(r) = \frac{1}{(2\pi h)^{2n}} \int\limits_{l_1+l_2=r} \hat{a}(l_1)\hat{b}(l_2) e^{\frac{i\sigma(l_1,l_2)}{2h}} \, dl_1.$$

It suffices to show that $K(x,\xi) = e^{ihA(D)}(a(x,\xi)b(y,\eta))\big|_{\substack{y=x\\ \eta=\xi}}$. Call the right-hand side $\tilde{K}$, and let $z = (x,\xi),\ w = (y,\eta)$. Then, we have that

$$\begin{aligned}\tilde{K}(z) &= \frac{1}{(2\pi h)^{4n}} \int\limits_{\mathbb{R}^{2n}}\int\limits_{\mathbb{R}^{2n}} \hat{a}(l_1)\hat{b}(l_2)e^{\frac{i}{2h}\sigma(hD_z,hD_w)}e^{\frac{i}{h}(l_1(z)+l_2(w))}\big|_{z=w}\, dl_1dl_2 \\ &= \frac{1}{(2\pi h)^{4n}} \int\limits_{\mathbb{R}^{2n}}\int\limits_{\mathbb{R}^{2n}} \hat{a}(l_1)\hat{b}(l_2)e^{\frac{i}{h}(l_1(z)+l_2(w))+\frac{i}{2h}\sigma(l_1,l_2)}\, dl_1dl_2\end{aligned}$$

Taking the semiclassical Fourier transform,

$$\hat{\tilde{K}}(r) = \frac{1}{(2\pi h)^{2n}} \int\limits_{\mathbb{R}^{2n}}\int\limits_{\mathbb{R}^{2n}} \left(\frac{1}{(2\pi h)^{2n}} \int\limits_{\mathbb{R}^{2n}} e^{\frac{i}{h}(l_1+l_2-r)(z)}\, dz \right) e^{\frac{i}{2h}\sigma(l_1,l_2)}\hat{a}(l_1)\hat{b}(l_2)\, dl_1dl_2,$$

and since the term in parenthesis equals $\delta_{l_1+l_2=r}$ in $\mathcal{S}'$, we have that $\tilde{K} = K$. □

It is useful to obtain asymptotic expansions of $a\#b$.

Proposition 3.9. *If $a, b \in \mathcal{S}$, then*

(1) *For $N = 0, 1, \cdots$,*

$$a\#b(x,\xi) = \sum_{k=0}^{N} \frac{i^k h^k}{k!} A(D)^k (a(x,\xi)b(y,\eta))\big|_{\substack{y=x\\ \eta=\xi}} + \mathcal{O}_{\mathcal{S}}(h^{N+1})$$

as $h \to 0$.

(2)

$$a\#b = ab + \frac{h}{2i}\{a,b\} + \mathcal{O}_{\mathcal{S}}(h^2),$$

and

$$[a^w(x,hD), b^w(x,hD)] = \frac{h}{i}\{a,b\}^w(x,hD) + \mathcal{O}_{\mathcal{S}}(h^3),$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket.

(3) *If* $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$, $a\#b = \mathcal{O}(h^\infty)$.

Here, the notation $a = \mathcal{O}_{\mathcal{S}}(h^k)$ as $h \to 0$ means that $p_{\alpha\beta}(a) \le C_{\alpha\beta}h^k$ as $h \to 0$, where $p_{\alpha\beta}$ denote the seminorms on $\mathcal{S}$ generating its Fréchet topology ($p_{\alpha\beta}(f) = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)|$). Additionally, $a = \mathcal{O}_{\mathcal{S}}(h^\infty)$ as $h \to 0$ means that there exists $h_0 > 0$ so that for all $N \in \mathbb{N}$, there exists $C_N \ge 0$ so that $p_{\alpha\beta}(a) \le C_N h^N$ for all $h \in (0, h_0)$. Note that (2) in the above proposition demonstrates mathematically that the Poisson bracket in the setting of classical mechanics is analogous to the commutator in the setting of quantum mechanics.

Proof. (1) This follows relatively closely from applying stationary phase to σ, where one obtains that

$$\int\limits_{\mathbb{R}^{2n}}\int\limits_{\mathbb{R}^{2n}} e^{\frac{i}{h}\sigma(z,w)}a(z,w)\, dzdw = (2\pi h)^{2n}\left(\sum_{k=0}^{N-1} \frac{h^k}{k!}\left(\frac{\sigma(D_z,D_w)}{i}\right)^k a(0,0) + \mathcal{O}(h^N) \right)$$

for $N \in \mathbb{N}$ and $a \in \mathcal{D}(\mathbb{R}^{2n})$. There are a few details, though. First, one must replace h by $h/2$ and σ by $-\sigma$. Next, one must be show that the remainders are of the correct form (we want, $\mathcal{O}_{\mathcal{S}}(h^{n+1})$, but stationary phase will only guarantee $\mathcal{O}(h^{n+1})$). One must mirror the proof of stationary phase, using Taylor's theorem on $e^{ihA(D)}$.

(2) We compute directly that

$$
\begin{aligned}
a\#b &= ab + ihA(D)(a(x,\xi)b(y,\eta))\big|_{\substack{y=x\\ \eta=\xi}} + \mathcal{O}_{\mathcal{S}}(h^2)\\
&= ab + \frac{ih}{2}(D_\xi a D_y b - D_x a D_\eta b)\big|_{\substack{y=x\\ \eta=\xi}} + \mathcal{O}_{\mathcal{S}}(h^2)\\
&= ab + \frac{h}{2i}\left(\nabla_\xi a\cdot\nabla_x b - \nabla_x a\cdot\nabla_\xi b\right) + \mathcal{O}_{\mathcal{S}}(h^2) = ab + \frac{h}{2i}\{a,b\} + \mathcal{O}_{\mathcal{S}}(h^2),
\end{aligned}
$$

and

$$
\begin{aligned}
[a^w, b^w] &= a^w b^w - b^w a^w = (a\#b - b\#a)^w\\
&= \left(ab + \frac{h}{2i}\{a,b\} + \frac{1}{2}h^2 A(D)^2(ab)\big|_{\substack{y=x\\ \eta=\xi}} - ba - \frac{h}{2i}\{b,a\} - \frac{1}{2}h^2 A(D)^2(ba)\big|_{\substack{y=x\\ \eta=\xi}} + \mathcal{O}_{\mathcal{S}}(h^3)\right)^w\\
&= \frac{h}{i}\{a,b\}^w + \mathcal{O}_{\mathcal{S}}(h^3),
\end{aligned}
$$

since the Poisson bracket is anticommutative.

(3) This follows immediately from the semiclassical expansion, since every term in the sum vanishes.

□

4. Symbol Classes

We would like to quantize symbols which are not Schwartz, such as polynomials. We will start with the definition of an *order function*, which is a function with, at most, polynomial growth which is comparable to itself.

Definition 2. A measurable function $m : \mathbb{R}^{2n} \to (0, \infty)$ is called an order function if there exists C, N so that

$$m(w) \le C\langle z - w\rangle^N m(z)$$

for all $w, z \in \mathbb{R}^{2n}$.

Here $\langle z\rangle := (1 + |z|^2)^{1/2}$ is the Japanese bracket of z. Now, we define some symbol classes for an order function m, with no restriction on regularity. Note that these are different from the "standard" Kohn-Nirenberg symbol classes.

$$
\begin{aligned}
S_\delta(m) &= \{a(x,\xi;h) \in C^\infty(\mathbb{R}^{2n}\times(0,1]) :\ |\partial_x^\alpha \partial_\xi^\beta a| \le C_{\alpha\beta} h^{-\delta|\alpha|} h^{-\delta|\beta|} m(x,\xi)\}\\
&= \{a(x,\xi;h) \in C^\infty(\mathbb{R}^{2n}\times(0,1]) :\ |\partial^\alpha a| \le C_\alpha h^{-\delta|\alpha|} m(x,\xi)\}.
\end{aligned}
$$

If $\delta = 0$, then we write $S_0(m) = S(m)$. Symbols $a \in S_\delta(m)$ can depend on h (in such a case, the constants must be independent of h), but we will often not mark this with our notation. If $m = 1$, then we will omit the m in our notation.

Note further that $S_\delta(m)$ is a Fréchet space, with the obvious semi-norm topology $\{p_j\}$ given by the infimum of the $C_{\alpha\beta}$'s, where $|\alpha| + |\beta| \le j$, or equivalently that

$$p_j(a) = \sup_{|\alpha+\beta|\le j}\ \sup_{x,\xi} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| h^{\delta(|\alpha|+|\beta|)} \frac{1}{m(x,\xi)}.$$

In fact, $\mathcal{S} \subset S(m)$ is dense in $S(m)$ in the $S((x\cdot\xi)^\epsilon m)$ topology, for any $\epsilon > 0$.

Remark 4.1. Let $a \in S_\delta$, and a_h denote a in the standard rescaling (i.e. the scaling that makes $h = 1$ in the Weyl quantization). Then,

$$|\partial^\alpha a_h| = h^{\frac{1}{2}|\alpha|}|\partial^\alpha a| \le C_\alpha h^{|\alpha|\left(\frac{1}{2}-\delta\right)}$$

for any multi-index α. If $\delta > 1/2$, then we have blow-up as $h \to 0$, so we only consider $\delta \in [0, 1/2]$. We call $\delta = 1/2$ the critical value of δ, as the right-hand side does not decay as $h \to 0$.

The formula for the Weyl quantization remains the same, and the introduced symbol classes allow us to define a^w as linear operators on $\mathcal{S}$ and $\mathcal{S}'$.

Proposition 4.2. *If $a \in S_\delta(m)$, then $a^w(x, hD) \in \mathcal{L}(\mathcal{S})$, and $a^w(x, hD) \in \mathcal{L}(\mathcal{S}')$.*

The idea of the first mapping property is to note that $L_1 = \frac{1-\xi D_y}{1+|\xi|^2}$ and $L_2 = \frac{1+(x-y)D_\xi}{1+|x-y|^2}$ are the identity operator on $e^{i(x-y)\cdot\xi}$ and use this to integrate by parts to get that $a^w(x, D)$, $x^\alpha a^w(x, D)u$, $\partial^\beta a^w(x, D)$, $x^\alpha \partial^\beta a^w(x, D) : \mathcal{S} \to L^\infty$. The continuity follows from similar arguments.

To get the second mapping property, one notes that for $u, v \in \mathcal{S}$, then $\langle a^w(x, D)u, v\rangle = \langle u, \tilde{a}^w(x, D)v\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the distributional pairing and $\tilde{a}(x, \xi) = a(x, -\xi)$. By the first part, $\tilde{a}^w(x, D)v \in \mathcal{S}$, and so $a^w(x, D)u$ is defined for $u \in \mathcal{S}'$. Continuity follows directly from the continuity in $\mathcal{S}$, as $\mathcal{S}'$ is a sequential space.

Composition extends to a similar result, where the remainder is in the appropriate symbol class.

Proposition 4.3. *If $a \in S_\delta(m_1)$ and $b \in S_\delta(m_1)$, then $a\#b \in S_\delta(m_1 m_2)$, and*

$$a^w(x, hD)b^w(x, hD) = (a\#b)^w(x, hD),$$

as operators on $\mathcal{S}$. Further,

$$a\#b = ab + \frac{h}{2i}\{a.b\} + \mathcal{O}_{S_\delta(m_1 m_2)}(h^{1-2\delta}),$$

and

$$[a^w(x, hD), b^w(x, hD)] = \frac{h}{i}\{a, b\}^w(x, hD) + \mathcal{O}_{S_\delta(m_1 m_2)}(h^{3-6\delta}).$$

Here, we say that $a = \mathcal{O}_{S_\delta(m)}(h^n)$ if

$$|\partial^\alpha a| \le C_{\alpha,n} h^{n-\delta|\alpha|} m,$$

for any α. The remainder in the expansion is due to the fact that the Poisson bracket terms involve a derivative in both x and ξ, leading to a loss of $h^{-2\delta}$, making the overall term of the order $h^{1-2\delta}$. Once again, we see that $\delta > 1/2$ yields blow-up as $h \to 0$, and $\delta = 1/2$ does not guarantee decay as $h \to 0$.

5. The Weyl Quantization on L^2

Thus far, we have only discussed the Weyl quantization acting on $\mathcal{S}$ or $\mathcal{S}'$. The goal in this section is to extend this to a bounded operator on L^2. First, we will consider the case of $a \in \mathcal{S}$, before moving on to more general symbol classes.

Proposition 5.1. *If $a \in \mathcal{S}$, then*

$$a^w(x, hD) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is bounded.

Proof. Recall that the Schwartz kernel of $a^w(x,hD)u(x)$ is

$$K(x,y) = \mathcal{F}^{-1}\left(a\left(\frac{x+y}{2},\cdot\right)\right)(x-y).$$

Since $a \in \mathcal{S}$, K is absolutely integrable in x for all y, and in y for all x. By Schur's lemma (Proposition 5.1 in [5]), $a^w(x,hD)$ is a bounded operator on L^2 (in fact, this bound is independent of h). □

We need a general result from functional analysis before proceeding, the *Cotlar-Stein Lemma.*

Theorem 5.2 (Cotlar-Stein Lemma). *Let H_1, H_2 be Hilbert spaces, and $A_j \in \mathcal{L}(H_1,H_2)$, for $j = 1, 2, \cdots$. Suppose that both*

$$\sup_j \sum_{k=1}^{\infty} \left\|A_j^* A_k\right\|^{1/2} \le C$$

and

$$\sup_j \sum_{k=1}^{\infty} \|A_j A_k^*\|^{1/2} \le C.$$

Then,

$$A := \sum_{j=1}^{\infty} A_j$$

converges in the strong operator topology, and $\|A\| \le C$.

We will relegate a proof of the Cotlar-Stein to the appendix. For general L^2 boundedness, we will only provide a sketch, as the full proof is fairly long and detailed; see e.g. [3],[4], or [7] for complete proofs (with different approaches).

Theorem 5.3 (Calderon-Vaillancourt). *Let $a \in S_\delta$, with $0 \le \delta \le 1/2$. Then,*

(1) *$a^w(x,D) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bounded.*
(2) *There exists M so that*

$$\|a^w(x,hD)\|_{L^2\to L^2} \le C \sum_{|\alpha|\le Mn} \|\partial^\alpha a\|_{L^\infty} h^{|\alpha|/2}.$$

Note that, in general, an operator A is bounded if and only if A^* is bounded, which is true if and only if A^*A is bounded. Thus, it suffices to show that $(a^w)^* a^w$ is bounded in L^2.

Proof. First, we start by creating a partition of unity that microlocalizes near lattice points. Select a smooth cutoff χ on $\mathbb{R}^{2n}$ such that $0 \le \chi \le 1$, $\chi = 0$ outside $B(0,2)$, and $\sum_{\alpha\in\mathbb{Z}^{2n}} \chi_\alpha = 1$, where $\chi_\alpha = \chi(\cdot - \alpha)$. Then, we can write

$$a = \sum_{\alpha\in\mathbb{Z}^{2n}} a_\alpha,$$

where $a_\alpha = \chi_\alpha a$. So, $a^w(x,hD) = \sum_{\alpha\in\mathbb{Z}^{2n}} a_\alpha^w(x,hD)$, and also

$$(a^w(x,hD))^* a^w(x,hD) = \sum_{\alpha,\beta\in\mathbb{Z}^{2n}} (a_\beta^w(x,hD))^*(a_\alpha^w(x,hD)) = \sum_{\alpha,\beta\in\mathbb{Z}^{2n}} (\bar{a}_\beta \# a_\alpha)^w.$$

Proving this is bounded in L^2 will complete the proof, which will follow from an application of the Cotlar-Stein lemma. Thus, we must show that

$$\sup_\alpha \sum_\beta \|(\bar{a}_\beta \# a_\alpha)^w\|^{1/2} \leq C.$$

Note that supp $a_\alpha \cap$ supp $a_\beta = \emptyset$ is $\alpha - \beta$ is large enough. In fact, one can show through careful integral splitting and support attention that $\bar{a}_\beta \# a_\alpha = \mathcal{O}(\langle \alpha - \beta \rangle^{-N})$, and then $\bar{a}_\beta \# a_\alpha$ is in $\mathcal{S}$ in (x, ξ) and $\alpha - \beta$. In the same way as Schur's inequality earlier, we get that

$$\|(\bar{a}_\alpha \# a_\beta)^w\|_{L^2 \to L^2}^{1/2} \leq C_N \langle \alpha - \beta \rangle^{-N/2}.$$

This is summable and gives one bound in the hypotheses of the Cotlar-Stein lemma. The other follows similarly, yielding the desired result.

The operator norm bound of a^w follows similarly to computing the error estimates in the asymptotic expansion of $a \# b$. □

Corollary 5.4. *If $a, b \in S_\delta$, with $0 \leq \delta < 1/2$, then*

$$(ab)^w(x, hD) = a^w(x, hD) b^w(x, hD) + \mathcal{O}_{L^2 \to L^2}(h^{1-2\delta})$$

as $h \to 0$.

Proof. Proposition 4.3 yields that

$$a \# b - ab = \mathcal{O}_{S_\delta}(h^{1-2\delta}).$$

The previous theorem implies further that

$$a^w b^w - (ab)^w = (a \# b - ab)^w = \mathcal{O}_{L^2 \to L^2}(h^{1-2\delta}).$$

□

If $a, b \in S_{1/2}$, and they have disjoint supports, then

$$a^w(x, hD) b^w(x, hD) = \mathcal{O}_{L^2 \to L^2}(h^\infty).$$

6. Compactness of the Weyl Quantization

Certain symbols make a^w into a compact operator on L^2. In view of the fact that a^w is self-adjoint on L^2, the spectral theorem yields that a^w has an othonormal basis of eigenfunctions of L^2. The specrtrum is real, and every non-zero element of the spectrum is an eigenvalue, with zero being the only accumulation point of the spectrum.

A starting point is for $a \in \mathcal{S}$.

Theorem 6.1. *If $a \in \mathcal{S}$, then*

$$a^w(x, D) \in \mathcal{K}(L^2(\mathbb{R}^n)).$$

Proof. Fix a bounded set $F \subset L^2$. We must show that there exists a bounded sequence (f_k) so that $a^w f_k$ converges in L^2. Note that since the Schwartz kernel of a^w, call it K, is an element of $\mathcal{S}(\mathbb{R}^{2n})$, we have that for any multi-indices α, β that

$$p_{\alpha\beta}(a^w u) \leq \sup_{(x,y)} |x^\alpha \partial_x^\beta \langle y \rangle^N K(x, y)| \int_{\mathbb{R}^n} \langle y \rangle^{-N} |u(y)| \, dy \leq C_{\alpha\beta} \|u\|_{L^2},$$

where $N > n/2$ and $p_{\alpha\beta}$ denotes the Schwartz seminorm corresponding to α and β, as defined earlier. We will need this later on. Call the above estimate $(*)$.

It will suffice to show that $g_k(x) := \langle x\rangle^N a^w(x,D)f_k(x)$ converges in L^∞, for $N > n/2$, since

$$\|a^w f_k - a^w f_j\|_{L^2} \le \left\|\langle x\rangle^{-N}\right\|_{L^2} \|g_k - g_j\|_{L^\infty}.$$

This is proved in a manner entirely analogous to the proof of the Arzela-Ascoli theorem. It is a fun argument, so we will go through it. First, let $\mathcal{A} = \{x_1, x_2, \cdots\}$ be a countable, dense subset of $\mathbb{R}^n$. Our earlier estimate $(*)$ shows that if $f_k \in F$, then the corresponding g_k is uniformly bounded. Since this a bounded sequence, $g_k(x_1)$ is bounded in $\mathbb{C}$, so it has a convergent subsequence $g_{1,k}(x_1)$. Similarly, the sequence $g_{1,k}(x_2)$ is bounded in $\mathbb{R}^n$, so it has a convergent subsequence $g_{2,k}$. Continuing, we define $h_k = g_{k,k}$, which is a subsequence of each $g_{n,k}$, and $g_{n,k}(x) = \langle x\rangle^N a^w(x,D)f_{n,k}(x)$. Let us re-notate this as g_k. Further, this converges pointwise on each $x_j \in \mathcal{A}$.

We want to show this converges in L^∞, so we prove that the sequence is Cauchy. Estimate $(*)$ shows that there exists M so that

$$|Dg_k(x)| \le M/3$$

and

$$\langle x\rangle |g_k(x)| \le M/2$$

(here, ∂ denotes the derivative). Let $\epsilon > 0$, and choose δ so that $M/\delta < \epsilon$. Cover $B(0,\delta)$ by balls $B(x_j, \epsilon/M)$, with $x_j \in \mathcal{A}$, which we can make finite by compactness (say the points are y_j, with j ranging from 1 to J). Since each sequence $g_\ell(x_j)$ is Cauchy, there exists N_2 so that for any $1 \le j \le J$,

$$|g_n(x_j) - g_m(x_j)| < \epsilon/3,$$

whenever $n, m > N_2$. Clearly,

$$\|g_n - g_m\|_{L^\infty(\mathbb{R}^n)} \le \max\{\|g_n - g_m\|_{L^\infty(|x|<\delta)}, \|g_n - g_m\|_{L^\infty(|x|\ge\delta)}\}.$$

The second term in the max satisfies the bound

$$\|g_n - g_m\|_{L^\infty(|x|\ge\delta)} \le 2\delta^{-1} \|\langle x\rangle g_n\|_{L^\infty(|x|\ge\delta)} < \epsilon.$$

On the other term in the max, given any $x \in B(0,\delta)$, there exists x_j so that $|x - x_j| < \epsilon/M$ from our covering, in which case we have

$$\begin{aligned} |g_n(x) - g_m(x)| &\le |g_n(x) - g_n(x_j)| + |g_n(x_j) - g_m(x_j)| \le |g_m(x_j) - g_m(x)| \\ &\le (\sup|Dg_n|)|x - x_j| + |g_n(x_j) - g_m(x_j)| + (\sup|Dg_m|)|x - x_j| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

using the mean-value inequality (and the convexity of the ball). The smallness of the first and third term came from our earlier estimate on the derivative of g_k, and the middle by the pointwise Cauchiness of g_k on points of $\mathcal{A}$. Since this is true for any $x \in B(0,\delta)$, we have that, in particular, (g_n) is Cauchy in L^∞. By completeness, $g_k \to g$ in L^∞ and hence $a^w f_k \to \langle x\rangle^{-N} g$ in L^2. □

This can be made more general. First, we will state a lemma, which is proved in a similar manner to L^2 boundedness.

Lemma 6.2. *If $a \in S(m)$, then for any $N \in \mathbb{N}$, there exists $C_N > 0$ so that*

$$\left\|b^w_{\alpha\beta}\right\|_{L^2\to L^2} \le C_N m(\alpha) m(\beta) \langle \alpha - \beta\rangle^{-N},$$

for all $\alpha, \beta \in \mathbb{Z}^{2n}$.

Here, we are using the same notation as in the L^2 boundedness section.

Theorem 6.3. *Suppose that $a \in S(m)$, and*

$$\lim_{\|(x,\xi)\|\to\infty} m(x,\xi) = 0.$$

Then,

$$a^w(x,D) \in \mathcal{K}(L^2(\mathbb{R}^n)).$$

Proof. Define $a_M^w = \sum_{|\alpha|<M} a_\alpha^w$. Since each $a_\alpha \in \mathcal{S}$, a_α^w compact, and so a_M^w is compact, as well (via Lemma 5.2). Since the operator norm limit of a sequence of compact operators is compact, it suffices to show that

$$\lim_{M\to\infty} \|a^w - a_M^w\|_{L^2\to L^2} = 0.$$

Write $a^w - a_M^w = \sum_{|\alpha|\geq M} a_\alpha^w$. By the Cotlar-Stein lemma,

$$\|a^w - a_M^w\| \leq \max\left\{ \sup_{|\alpha|\geq M} \sum_{|\beta|\geq M} \left\|a_\alpha^w (a_\beta^w)^*\right\|^{1/2}, \sup_{|\alpha|\geq M} \sum_{|\beta|\geq M} \left\|(a_\alpha^w)^* a_\beta^w\right\|^{1/2} \right\}.$$

Now, $(a_\alpha^w)^* a_\beta^w = (\bar{a}_\alpha \# a_\beta)^w = b_{\alpha\beta}^w$, so applying the previous lemma provides that

$$\begin{aligned} \sup_{|\alpha|\geq M} \sum_{|\beta|\geq M} \left\|(a_\alpha^w)^* a_\beta^w\right\|^{1/2} &\leq C_N \sup_{|\alpha|\geq M} \sum_{|\beta|\geq M} \sqrt{m(\alpha)m(\beta)}\langle \alpha-\beta\rangle^{-N/2} \\ &\leq C \sup_{|\alpha|\geq M} m(\alpha). \end{aligned}$$

The other case is similar. Thus,

$$\|a_w - a_M^w\| \leq C \sup_{|\alpha|\geq M} m(\alpha) \to 0$$

as $M \to \infty$, since m decays to 0 as infinity, by assumption.

□

Remark 6.4. The converse also holds: if $a \in S(m)$ and a^w is compact, then,

$$\lim_{\|(x,\xi)\|\to\infty} m(x,\xi) = 0.$$

7. Ellipticity and the Gårding Inequality

Before discussing the Gårding inequality, we will discuss invertibility. One might wonder is a non-vanishing symbol gives rise to an invertible quantization.

Definition 3. We say a is *elliptic* if there exists an h-independent constant $\gamma > 0$ so that $|a| \geq \gamma > 0$ on $\mathbb{R}^{2n}$. More generally, we say that a is *elliptic in $S(m)$* if for some constant $\gamma > 0$, we have $|a| \geq \gamma m$.

Theorem 7.1. *Suppose $a \in S_\delta(m)$, for some $\delta \in [0, 1/2)$, is elliptic in $S(m)$.*

1. *If $m = 1$, then there exists $h_0 > 0$ so that $a^w(x,hD)^{-1}$ exists as a bounded operator on $L^2(\mathbb{R}^n)$, for $0 < h < h_0$.*
2. *If $m \geq 1$, then there exists $h_0 > 0$ and $C > 0$ so that*

$$\|a^w(x,hD)u\|_{L^2} \geq C\|u\|_{L^2}$$

for any $u \in \mathcal{S}$ and $0 < h \leq h_0$

Before proving this, we state the following elementary proposition.

Proposition 7.2. *Let X, Y be Banach spaces and $A \in \mathcal{L}(X, Y)$. Suppose that there exist $B_1, B_2 \in \mathcal{L}(Y, X)$ so that*

$$AB_1 = I + R_1 \quad \text{on } Y$$

and

$$B_2 A = I + R_2 \quad \text{on } X,$$

with $\|R_j\| < 1$. Then, A is invertible.

Proof. Since $\|R_j\| < 1$, $I + R_j$ is invertible. Calling $C_1 = B_1(I + R_1)^{-1}$ and $C_2 = (I + R_2)^{-1}B_2$, we have that $AC_1 = I$ and $C_2 A = I$, and so $A^{-1} = C_1 = C_2$. $\square$

Proof of Theorem (7.1). Note $b = 1/a \in S_\delta(1/m)$, and so using the composition formula,

$$a\#b = 1 + r_1, \qquad b\#a = 1 + r_2,$$

with $r_j \in h^{1-2\delta} S_\delta$. If we set $A = a^w$, $B = b^w$, $R_j = r_j^w$, then we have

$$AB = I + R_1$$

and

$$BA = I + R_2,$$

with

$$\|R_j\|_{L^2 \to L^2} = \mathcal{O}(h^{1-2\delta}) \leq 1/2 < 1,$$

for small enough h_0 and $h \in (0, h_0]$. If $m = 1$, then the result follows from L^2 boundedness and the previous proposition. If $m \geq 1$, then for $u \in \mathcal{S}$, we have

$$\|u\|_{L^2} = \left\|(I + R_2)^{-1} b^w a^w u\right\|_{L^2} \leq \left\|(I + R_2)^{-1}\right\|_{L^2} \|b^w\|_{L^2} \|a^w u\| \leq C \|a^w u\|_{L^2},$$

using that $b \in S(1/m) \subset S$, so b^w is bounded in L^2. $\square$

One can utilize similar arguments to construct *elliptic parametrices* up to order $\mathcal{O}(h^\infty)$. In particular,

Proposition 7.3. *Let $a \in S(m_1)$ and $p \in S(m_2)$ be elliptic on the support of a. Then, there exists $q \in S(m_1/m_2)$ such that*

$$p\#q = a + \mathcal{O}(h^\infty)_{S_{m_1}}$$
$$q\#p = a + \mathcal{O}(h^\infty)_{S_{m_1}}.$$

We will not prove this, but it is proved via repeated division by p. Using this result one can establish elliptic estimates of the form

$$\|a^w u\|_{L^2} \leq \|p^w u\|_{L^2} + \mathcal{O}(h^\infty) \|u\|_{L^2}$$

whenever $p \in S(m)$ is elliptic on the support of $a \in S(1)$, which we will also omit.

Now, we move onto the Gårding inequality. We will start with a weak version.

Theorem 7.4. *Let $a \in S$ be real-valued, and $a \geq \gamma > 0$ on $\mathbb{R}^{2n}$. Then, for any $\epsilon > 0$, there exists $h_0 > 0$ so that*

$$(a^w(x, hD)u, u)_{L^2(\mathbb{R}^n)} \geq (\gamma - \epsilon) \|u\|^2_{L^2(\mathbb{R}^n)}$$

for all $0 < h \leq h_0$ and $u \in L^2(\mathbb{R}^n)$.

This states, roughly, that if $a \geq \gamma$, then $a^w \geq \gamma - \epsilon$, as an operator.

Before proving this result, we recall a spectral result from functional analysis.

Theorem 7.5. *Let H be a Hilbert space, and $A \in \mathcal{K}(H)$ be self-adjoint. If $\sigma(A) \subset [a, \infty)$, then*

$$(Au, u)_H \geq a \|u\|^2$$

for all $u \in H$.

This result follows from the method of proving that the spectrum of a bounded, self-adjoint operator is real.

Proof of Theorem (7.4). Let $\lambda < \gamma - \epsilon$, allowing us to validly define $b = (a - \lambda)^{-1}$. Then,

$$(a - \lambda)\#b = 1 + \frac{h}{2i}\{a - \lambda, b\} + \mathcal{O}_S(h^2) = 1 + \mathcal{O}_S(h^2),$$

since the Poisson bracket vanishes (as a consequence of the inverse function theorem). Thus, we have

$$(a^w - \lambda)b^w = I + \mathcal{O}_{L^2 \to L^2}(h^2).$$

We get a similar expression for $b^w(a^w - \lambda)$, so Proposition 7.2 yields that $a^w - \lambda$ is invertible, provided $\lambda < \gamma - \epsilon$, so $\rho(a^w) \supset (-\infty, \gamma - \epsilon)$. Then, $\sigma(a^w) \subset [\gamma - \epsilon, \infty)$. Combining this with the fact that a^w is self-adjoint on L^2, it follows that

$$(a^w u, u)_{L^2} \geq (\gamma - \epsilon) \|u\|_{L^2}^2 .$$

□

We can make this result quite a bit sharper.

Theorem 7.6. *Let $a \in S$ be non-negative on $\mathbb{R}^{2n}$. Then, there exist constants $C \geq 0$, $h_0 > 0$ so that*

$$(a^w(x, hD), u, u)_{L^2(\mathbb{R}^n)} \geq -Ch \|u\|_{L^2(\mathbb{R}^n)}^2 ,$$

for all $0 < h < h_0$ and $u \in L^2(\mathbb{R}^n)$.

This says that, roughly, if $a \geq 0$, then (as an operator) $a^w \geq 0 +$ lower order terms.

Remark 7.7. This is true for any quantization $Op_t(a)$, with $t \in [0, 1]$. In fact, we have the *Fefferman-Phong inequality* for the Weyl quantization:

$$(a^w u, u)_{L^2} \geq -Ch^2 \|u\|_{L^2}^2 .$$

First, we need an advanced calculus estimate:

Lemma 7.8. *If $f : \mathbb{R}^n \to \mathbb{R}$ is non-negative and C^2 with bounded Hessian $|D^2 f| \leq A$, then*

$$|Df| \leq (2Af)^{1/2}.$$

This is a direct consequent of Taylor's theorem.

Proof of Theorem (7.6). We would like to proceed as in the proof of Theorem 7.4. Let $\tilde{h}$ be sufficiently small, and let $\lambda = h/\tilde{h}$. We want to show that $h(a+\lambda)^{-1} \in \tilde{h}S_{1/2}$, where we say $b \in \tilde{h}S^{1/2}$ if

$$|\partial^\alpha b| \leq C_\alpha h^{-\alpha|/2} \tilde{h},$$

for all α, with C_α independent of both h and $\tilde{h}$. First, we note that there exist $C_{\beta_1, \cdots, \beta_k}$ such that

$$\partial^\alpha (a + \lambda)^{-1} = (a + \lambda)^{-1} \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha = \beta_1 + \cdots + \beta_k \\ |\beta_j| \geq 1}} C_{\beta_1, \cdots \beta_k} \prod_{j=1}^{k} ((a + \lambda)^{-1} \partial^{\beta_j} a).$$

This can be proven inductively, but it is messy and hence omitted. By the previous lemma,

$$\lambda^{1/2}|Da| \le Ca^{1/2}\lambda^{1/2} \le C(a+\lambda).$$

In particular, if $|\beta| = 1$, then

$$|\partial^\beta a|(a+\lambda)^{-1} \le C\lambda^{-1/2}.$$

Also, since $a \in S$, there exists C so that

$$|\partial^\beta a|(a+\lambda)^{-1} \le C\lambda^{-1}$$

for any $|\beta| \ge 2$ (this constant depends on β).

Now, if we take restrict $\lambda \in (0,1]$, we have for any $\alpha = \beta_1 + \cdots + \beta_k$ that

$$\begin{aligned}\left|\prod_{j=1}^k (a+\lambda)^{-1}\partial^{\beta_j} a\right| &= \left(\left|\prod_{|\beta_j|=1} (a+\lambda)^{-1}\partial^{\beta_j} a\right|\right)\left(\left|\prod_{2\le|\beta_j|\le k} (a+\lambda)^{-1}\partial^{\beta_j} a\right|\right)\\ &\le C\prod_{|\beta_j|=1}\lambda^{-1/2}\prod_{2\le|\beta_j|\le k}\lambda^{-1} \le C\prod_{j=1}^k \lambda^{-|\beta_j|/2}\\ &= C\lambda^{-|\alpha|/2}.\end{aligned}$$

Thus,

$$|\partial^\alpha (a+\lambda)^{-1}| \le C_\alpha (a+\lambda)^{-1}\lambda^{-|\alpha|/2} = C_\alpha (a+\lambda)^{-1} h^{-|\alpha|/2}\tilde{h}^{|\alpha|/2},$$

and since $\tilde{h}$ is small and $\lambda \in (0,1]$, we have $h(a+\lambda)^{-1} \in \tilde{h}S_{1/2}$. Since $a \in S,\ a+\lambda \in S \subset S_{1/2}$. This allows us to define the #-composition of $a_\lambda := a+\lambda$ and $b_\lambda := (a+\lambda)^{-1}$. By Taylor's theorem (with integral remainder) and the definition of composition,

$$\begin{aligned}a_\lambda \# b_\lambda(z) &= e^{ihA(D)} a_\lambda(z) b_\lambda(w)\big|_{z=w}\\ &= 1 + \int_0^1 (1-t)e^{ithA(D)}(ihA(D))^2 (a_\lambda(z)b_\lambda(w))\big|_{w=z}\, dt\\ &= 1 + r_1(z),\end{aligned}$$

since (as before) the Poisson bracket vanishes. Following a similar process, we get $b_\lambda \# a_\lambda(z) = 1 + r_2(z)$. That is, we have $r_1,\ r_2$ so that

$$a_\lambda^w b_\lambda^w = I + r_1^w,$$

and

$$b_\lambda^w a_\lambda^w = I + r_2^w.$$

Since the Weyl quantization is bounded in L^2 for the above symbols, it only remains to show that $\left\|r_j^w\right\|_{L^2\to L^2} < 1$. The process is similar for both, so we only show r_1. Since $hb_\lambda \in \tilde{h}S_{1/2}$, it is obvious that $h^2\partial^\alpha b_\lambda \in \tilde{h}S_{1/2}$ for $|\alpha| = 2$. Owing to that $e^{ithA(D)}$ preserves $\tilde{h}S_{1/2}$, we obtain, using the L^2 boundedness of a_λ^w and b_λ^w that

$$\|r_1^w\|_{L^2\to L^2} \le C\tilde{h} \le 1/2,$$

for $\tilde{h}$ small enough. The result is similarly true for r_2^w, and by Proposition 7.2, it follows that $a^w + \lambda$ is invertible. Similarly, $a^w + \gamma + \lambda$ is invertible for $\gamma \ge 0$. Thus, $\sigma(a^w) \subset [-\lambda, \infty)$, and now our spectral theory result provides that

$$(a^w u, u)_{L^2} \ge -\lambda \left\|u\right\|_{L^2}^2,$$

for all $u \in L^2$. Recalling that $\lambda = h/\tilde{h}$, we are done.

$\square$

A. Proof of the Cotlar-Stein Lemma

Theorem A.1 (Cotlar-Stein Lemma)**.** *Let H_1, H_2 be Hilbert spaces, and $A_j \in \mathcal{L}(H_1, H_2)$, for $j = 1, 2, \cdots$. Suppose that both*

$$\sup_j \sum_{k=1}^{\infty} \left\| A_j^* A_k \right\|^{1/2} \leq C$$

and

$$\sup_j \sum_{k=1}^{\infty} \| A_j A_k^* \|^{1/2} \leq C.$$

Then,

$$A := \sum_{j=1}^{\infty} A_j$$

converges in the strong operator topology, and $\|A\| \leq C$.

We will prove this in a sequence of steps, generally following the procedure as in [1].

Proposition A.2. *The Cotlar-Stein Theorem holds when each sum is up to a finite n.*

Proof. Since

$$\|A\| = \|AA^*\|^{1/2} = \|A^*A\|^{1/2},$$

it follows that $\|A\|^{2N} = \left\|(AA^*)^N\right\|$, for any $N \in \mathbb{N}$. By the triangle inequality,

$$\left\|(AA^*)^N\right\| \leq \sum_{i_1, j_1, \cdots i_N, j_N \in \{1, \cdots, n\}} \left\| A_{i_1} A_{j_1}^* \cdots A_{i_N} A_{j_N}^* \right\|$$

The above can be bounded by both

$$\sum_{i_1, j_1, \cdots i_N, j_N \in \{1, \cdots, n\}} \left\| A_{i_1} A_{j_1}^* \right\| \cdots \left\| A_{i_N} A_{j_N}^* \right\|$$

and

$$\sum_{i_1, j_1, \cdots i_N, j_N \in \{1, \cdots, n\}} \|A_{i_1}\| \left\| A_{j_1}^* A_{i_2} \right\| \cdots \left\| A_{j_{N-1}}^* A_{i_N} \right\| \left\| A_{j_N}^* \right\|$$

$$\leq \sum_{i_1, j_1, \cdots i_N, j_N \in \{1, \cdots, n\}} C^2 \left\| A_{j_1}^* A_{i_2} \right\| \cdots \left\| A_{j_{N-1}}^* A_{i_N} \right\|,$$

since it follows directly by our assumptions that $\|A_{i_1}\|, \left\|A_{j_N}^*\right\| \leq C$. Taking the geometric mean of the two bounds, we get that

$$\|A\|^{2N} \leq C \sum_{i_1, j_1, \cdots i_N, j_N \in \{1, \cdots, n\}} \left\| A_{i_1} A_{j_1}^* \right\|^{1/2} \left\| A_{j_1} A_{i_2}^* \right\|^{1/2} \cdots \left\| A_{j_{N-1}}^* A_{i_N} \right\|^{1/2} \left\| A_{i_N} A_{j_N}^* \right\|^{1/2} \leq nC^{2N},$$

by summing the series in $j_N, i_N, \cdots j_1, i_1$ and using our hypothesis along the way. Thus,

$$\|A\| \leq n^{1/2N} C,$$

and taking the limit as $N \to \infty$ completes the proof. $\square$

This is enough to establish weak operator convergence.

Corollary A.3. *$A = \sum_{j=1}^{\infty} A_j$ converges in the weak operator topology.*

Proof. We must show that for any $u \in H_1,\ v \in H_2$, the series $\sum_j \langle v, A_j u\rangle_2$ converges. Suppose not. Then, for any $M > 0$, there exists $I = \{i_1, \cdots, i_N\} \subset \mathbb{N}$ for which

$$\left|\sum_{j\in I} \langle v, A_j u\rangle_2\right| > M.$$

Call $A_I := \sum_{i\in I} A_i$. Since the above is true for any $M > 0$, there exists some finite I so that

$$\left|\sum_{i\in I} \langle v, A_i u\rangle_2\right| > C \left\|u\right\|_1 \left\|v\right\|_2.$$

On the other hand, using the previous lemma and the Cauchy-Schwarz inequality,

$$\left|\sum_{i\in I} \langle v, A_i u\rangle_2\right| = |\langle v, A_I u\rangle_2| \le \left\|v\right\|_2 \left\|A_I u\right\|_2 \le C \left\|u\right\|_1 \left\|v\right\|_2,$$

a contradiction. □

Lemma A.4. *Let $n \in \mathbb{N}$, and $\{I_j\}_{j=1}^n$ be finite subsets of $\mathbb{N}$, with $I_k \cap I_l = \emptyset$ if $k \neq l$. Then,*

$$\left\|\sum_{j=1}^n A_{I_j}^* A_{I_j}\right\| \le C^2,$$

where $A_{I_j} = \sum_{i\in I_j} A_i$.

Proof. Let $I = \bigcup_{j=1}^n I_j$, and $I^2 = I \times I$ Our sum to bound is now $\sum_{(i,j)\in I^2} A_i^* A_j$. Then, by the disjointness, it follows that for any $N \in \mathbb{N}$,

$$\left\|\left(\sum_{j=1}^n A_{I_j}^* A_{I_j}\right)^N\right\| \le \sum_{i_1, j_1, \cdots i_N, j_N \in I} \left\|T_{i_1} T_{j_1}^* \cdots T_{i_N} T_{j_N}^*\right\|.$$

By an analogous process to in the first lemma, we get that this is bounded above by $|I|C^{2N}$, where $|I|$ is the counting measure of I. If $N = 2^m$,

$$\left\|\sum_{j=1}^n A_{I_j}^* A_{I_j}\right\|^N = \left\|\left(\sum_{j=1}^n A_{I_j}^* A_{I_j}\right)^2\right\|^{N/2} = \cdots = \left\|\left(\sum_{j=1}^n A_{I_j}^* A_{I_j}\right)^N\right\| \le |I|C^{2N}.$$

Now, taking the limit as $N \to \infty$, we get the result. □

Proof of the Cotlar-Stein Lemma. Let $u \in H_1$, and set $v_j = A_j u$. We show that $\sum_{j=1}^{\infty} v_j$ converges in H_2. Suppose not. Then, there exist $\epsilon > 0$ and infinitely many disjoint I_j so that

$$\left\|\sum_{i\in I_j} v_i\right\|_2 \ge \epsilon.$$

Hence, $n \in \mathbb{N}$ exists so that

$$\sum_{j=1}^{n} \left\| \sum_{i \in I_j} v_i \right\|_2^2 > C^2 \|u\|_1^2 .$$

But, by the above lemma,

$$\begin{aligned}\sum_{j=1}^{n} \left\| \sum_{i \in I_j} v_i \right\|_2^2 &= \sum_{j=1}^{n} \langle A_{I_j} u, A_{I_j} u \rangle_2 = \langle u, \sum_{j=1}^{n} A_{I_j}^* A_{I_j} u \rangle_1 \\ &\leq \|u\|_1 \left\| \sum_{j=1}^{n} A_{I_j}^* A_{I_j} u \right\|_1 \leq C^2 \|u\|_1^2 ,\end{aligned}$$

a contradiction. □

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