AN INTRODUCTION TO BMO AND BMO^{-1} SPACES

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ABSTRACT. We give an brief overview of the space of bounded mean oscillations (BMO), followed by an introduction to the space BMO^{-1}. After providing a characterization of BMO^{-1} as the divergence of a BMO vector field, we will demonstrate their utility via an application to the incompressible Navier-Stokes equations, as in the seminal paper [7]. In particular, BMO^{-1} is the most general space of incompressible initial data currently known in which solutions of such equations are well-posed.

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1. INTRODUCTION

The space of bounded mean oscillations, or BMO, were first introduced by John and Nirenberg in [5]. This space of functions, whose elements must have limited oscillation from their average over any ball (or cube), arise as a natural substitute for the L^∞ space in a manner analogous to the Hardy space H^1 as a substitute for L^1 (in fact, a major result is that $\mathrm{BMO}(\mathbb{R}^n) = (H^1(\mathbb{R}^n))'$, see [10]). (To be a bit more precise on this "substitution" statement, the precise statement is that the image of L^∞ under a Calderón-Zygmund operator lies in BMO, although this will not be discussed.)

In [7], Koch and Tataru search for the largest function space for which initial data in such a space yields local and/or global solutions. First, they define a space BMO^{-1} whose elements are defined by having a finite norm related to the BMO space (semi)norm, then they characterize BMO^{-1} as tempered distributions which are the divergence of a BMO vector field. They proceed to prove existence of global solutions to the (incompressible) Navier-Stokes equations (in the appropriate function space) with divergence-free initial data with small BMO^{-1} norm, as well as a local analogue. Many other spaces for which existence results are known are imbedded in BMO^{-1}, and it is thought to (potentially) be the largest space for initial data to admit solutions (with certain desirable properties).

In this paper, we will start with a discussion of BMO and some essential properties that the space possesses, including a proof of the John-Nirenberg inequality, which gives an exponentially-decaying bound on the measure of values where a function differs from its average by a prescribed amount. From there, we will briefly consider a characterization of BMO using Carleson measures. Then, we will move on to discussing BMO^{-1}, including a proof of a key characterization of the space as the divergence of a BMO vector field. From there, the results of Koch and Tataru will be briefly explained, including a short summary the proof of their ground-breaking existence result. Finally, we will end with a discussion on the significance of BMO^{-1} to Navier-Stokes existence theory.

2. The Space of Bounded Mean Oscillations (BMO)

We will begin with a discussion of various fundamental properties of BMO before moving on to the BMO^{-1} space. BMO is defined as the space

$$\begin{aligned}\text{BMO}(\mathbb{R}^n) &= \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \ \frac{1}{m(B)} \int\limits_B |f(x) - f_B| \ dx \le A, \ \text{some } A \ge 0, \ \text{any ball } B \subset \mathbb{R}^n \right\} \\ &= \left\{ f \in L^1_{loc}(\mathbb{R}_n) : \ f^{\#} \in L^\infty(\mathbb{R}^n) \right\},\end{aligned}$$

where

$$f_B = \frac{1}{m(B)} \int\limits_B f(x) \ dx$$

is the average of f over B,

$$f^{\#}(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int\limits_{B_r(x)} |f(y) - f_{B_r(x)}| \ dy,$$

and $m(B)$ is the Lebesgue measure of B. Thus, BMO is the collection of locally integrable functions whose average oscillation is uniformly bounded over any ball. One naturally defines a seminorm on BMO as the smallest such A, or equivalently

$$\|f\|_{\text{BMO}} = \left\| f^{\#} \right\|_{L^\infty}.$$

In order to make this normed, one typically defines BMO modulo constants, since these would other constitute non-zero elements with zero norm. Quotienting out by constants forms BMO into a normed vector space.

The definition of BMO can be modified in various ways to provide equivalent characterizations. One could use cubes, instead of balls, as originally done in [5]. Additionally, one could replace f_B by a constant c_B, where c_B is chosen to minimize $f^{\#}$. Prototypical examples of elements of BMO are L^∞ functions, functions of the form $\log |p(x)|$ (where p is a polynomial), and the log of A_p weights (which is the class of weights ω for which the Hardy-Littlewood maximal operator is bounded in $L^p(d\omega)$). It is not difficult to see that the space is invariant under both translation and scaling.

A major result for BMO spaces is that it is the dual of the Hardy space: $\text{BMO} = (H^1(\mathbb{R}^n))'$, which implies that BMO is a Banach space. This fact is proved in [10]. It involves first showing that given $f \in \text{BMO}$, the linear functional ℓ defined on an appropriate dense subspace H^1_a (finite linear combinations of H^1 atoms) of H^1 given by

$$\ell(g) = \int\limits_{\mathbb{R}^n} f(x) g(x) \ dx,$$

with $g \in H^1_a$, has a unique bounded extension with $\|\ell\| \le c \|f\|_{\text{BMO}}$ $(< \infty)$. Then, one can prove that any continuous linear functional on H^1 takes the same form as ℓ, with $f \in \mathrm{BMO}$, and with the bound $\|f\|_{\mathrm{BMO}} \leq c' \|\ell\|$. While a significant result, this is too far of a digression from the direction we are headed, so the proof is omitted.

While it is clear that $L^\infty \subset \mathrm{BMO}$, it is also true that BMO functions are "nearly bounded," in the sense that they are limited in how far they are vary from their average. This is the content of the John-Nirenberg Inequality.

Theorem 2.1 (John-Nirenberg Inequality)**.** *If $f \in \mathrm{BMO}(\mathbb{R}^n)$. Then, there exist positive constants c_1 and c_2 (dependent on n) so that, for any $\alpha > 0$ and cube (or ball) Q,*

$$m\left(\{x \in Q : \ |f(x) - f_Q| > \alpha\}\right) \leq c_1 \exp\left(-\frac{c_2 \alpha}{\|f\|_{\mathrm{BMO}}}\right) m(Q).$$

This succinctly states a key property of BMO functions which allows us to intuitively understand their behavior (arguably more concretely than as continuous linear functionals on H^1, at least). There are various proofs of the result, such as in [10], where they use the previously-mentioned H^1 duality. We will instead proceed as in [8], first proving what is essentially a special/simple case of the Calderón-Zygmund decomposition theorem. This begins by proving a covering lemma.

Proposition 2.2. *Let $f \in L^1(Q)$, with $Q \subset \mathbb{R}^n$ a cube. If $\alpha > m(Q)^{-1} \|f\|_{L^1(Q)}$, then there exists a countable, disjoint collection $\{Q_j\}$ of subcubes for which*

(1) $|f(x)| \leq \alpha$ *for m-a.e. $x \in Q \setminus \bigcup_j Q_j$*

(2)
$$\alpha < \frac{1}{m(Q_j)} \int\limits_{Q_j} |f(x)| \, dx < 2^n \alpha$$

(3)
$$\sum_j m(Q_j) \leq \alpha^{-1} \|f\|_{L^1(Q)}$$

Proof of 2.2. First, we sub-divide Q into 2^n sub-cubes $\{Q_j'\}$, where each side of Q_j' has half of the original length. If $Q' \in \{Q_j'\}$ satisfies

$$\alpha \leq \frac{1}{m(Q')} \int\limits_{Q'} |f(x)| \, dx,$$

then we add Q' to a new collection $\{Q_j\}$. Now, we take the remaining cubes, and perform the same sub-division to each such cube, and repeat. Note that if $Q_j \in \{Q_j\}$, then it is contained in some unselected cube Q_j', which has double the side-length, and so

$$\frac{1}{m(Q_j')} \int\limits_{Q_j'} |f(x)| \, dx < \alpha,$$

which then gives that

$$\alpha \leq \frac{1}{m(Q_j)} \int\limits_{Q_j} |f(x)| \, dx = \frac{1}{2^n m(Q_j')} \int\limits_{Q_j} |f(x)| \, dx \leq \frac{1}{2^n m(Q_j')} \int\limits_{Q_j'} |f(x)| \, dx < 2^n \alpha.$$

In particular, this gives (2), and (3) follows directly from (2).

Next, let $x \in Q \setminus \bigcup_j Q_j$. Call $Q \setminus \bigcup_j Q_j =: Q' = \bigcap_j Q_j'$. Then, for every Q_j', we have by construction that

$$\frac{1}{m(Q_j')} \int_{Q_j'} |f(x)| \, dx < \alpha,$$

and since $Q_{j+1}' \subset Q_j'$, with the side-length going to 0 as $j \to \infty$, it follows from elementary maximal function theory (Lebesgue differentiation theorem), that since $f \in L^1(Q)$,

$$\frac{1}{m(Q_j')} \int_{Q_j'} |f(x)| \, dx \to f(x)$$

for m-a.e. $x \in Q'$, and thus $f(x) \le \alpha$ for m-a.e. $x \in Q'$. □

Corollary 2.3. *Let $\|f\|_{\mathrm{BMO}} \le 1$, and $Q \subset \mathbb{R}^n$ be a cube. Then, there exists a countable, disjoint collection of sub-cubes $\{Q_j\}$ such that*

$$\sum_j m(Q_j) \le \frac{1}{2} m(Q),$$

and for any $x \in Q$,

$$(f(x) - f_Q)\chi_Q(x) = g(x)\chi_Q(x) + \sum_j \left(f(x) - f_{Q_j}\right) \chi_{Q_j}(x),$$

with $|g(x)| \le C_n$, where $C_n \in \mathbb{R}^+$ depends only on n.

Proof of 2.3. Apply the previous proposition to $u = f - f_Q$, with $\alpha = 2$. Then, by taking

$$g(x) = \begin{cases} u(x) & x \in Q \setminus \bigcup_j Q_j \\ u_{Q_j} & x \in Q_j, \end{cases}$$

we immediately get that $|g(x)| \le 2^{n+1}$ for m -a.e. x,

$$\sum_j m(Q_j) \le \frac{1}{2} \|u\|_{L^1(Q)} \le \frac{1}{2} \|f\|_{\mathrm{BMO}} \, m(Q) \le \frac{1}{2} m(Q),$$

and since it is readily seen that

$$u(x)\chi_Q(x) = g(x)\chi_Q(x) + \sum_j \left(u(x) - u_{Q_j}\right) \chi_{Q_j}(x)$$

and $u - u_{Q_j} = f - f_{Q_j}$ on Q_j, the decomposition follows directly. □

Now, we can prove the John-Nirenberg inequality.

Proof of 2.1. By scaling, we can take $\|f\|_{\mathrm{BMO}} = 1$. Note that it will suffice to prove that if Q is a cube, then for any $k \in \mathbb{N}$,

$$m(\{x \in Q : \; |f(x) - f_Q| > 2^{n+1}k\}) \le 2^{-k} m(Q)$$

(it suffices to prove it for $\alpha \ge 2^{n+1}k$, in which case proving the above gives the result with $c_1 = e$, $c_2 = 2^{n+1}e^{-1}$).

First, we can use the previous corollary to write

$$(f - f_Q)\chi_Q = g\chi_Q + \sum_j (f - f_{Q_j})\chi_{Q_j},$$

with $|g| \leq 2^{n+1}$ and $\sum\limits_j m(Q_j) \leq \frac{1}{2}m(Q)$. Then, note that if

$$2^{n+1} < |f(x) - f_Q|,$$

we have

$$\begin{aligned} 2^{n+1} &< |g(x)| + \sum_j |f(x) - f_{Q_j}|\chi_{Q_j}(x) \\ &\leq 2^{n+1} + \sum_j |f(x) - f_{Q_j}|\chi_{Q_j}(x) \\ &\leq 2^{n+1} + \sup_x |f(x) - f_Q| \sum_j \chi_{Q_j}. \end{aligned}$$

Since the result is obvious if $f(x) = f_Q$, assuming to the contrary yields that for $k = 1$,

$$\begin{aligned} &m(\{x \in Q : \ |f(x) - f_Q| > 2^{n+1}\}) \\ &= m\left(\left\{x \in Q : \ |g\chi_Q(x) + \sum_j (f(x) - f_{Q_j})\chi_{Q_j}(x)| > 2^{n+1}\right\}\right) \\ &\leq m\left(\left\{x \in Q : \ \sum_j \chi_{Q_j}(x) > 0\right\}\right) = \sum_j m(Q_j) \\ &\leq \frac{1}{2}m(Q). \end{aligned}$$

This proves the $k = 1$ case. Next, apply the corollary to $f - f_{Q_j}$ on Q_j for each j to get

$$(f - f_Q)\chi_Q = g\chi_Q + \sum_j g_j\chi_{Q_j} + \sum_j (f - f_{Q'_j})\chi_{Q'_j},$$

with $|g_j| \leq 2^{n+1}$ and $\sum\limits_j m(Q'_j) \leq \frac{1}{4}m(Q)$. Proceeding as in the $k = 1$ case with our new decomposition provides the $k = 2$ case. Continuing inductively completes the proof. □

Note that we easily get the following corollary, stating that BMO functions are locally L^p for $p \in (1, \infty)$.

Corollary 2.4. *If $Q \subset \mathbb{R}^n$ is a cube and $f \in \mathrm{BMO}(\mathbb{R}^n)$, then*

$$\frac{1}{m(Q)} \int_Q |f(x) - f_Q|^p \, dx \leq C_{n,p} \|f\|_{\mathrm{BMO}}.$$

Proof of 2.4. Without loss of generality, let $\|f\|_{\mathrm{BMO}} = 1$. Define the distribution function of f to be the function $\lambda_f : [0, \infty) \to [0, \infty]$ given by

$$\lambda_f(\alpha) = m(\{x \in Q : \ |f(x)| > \alpha\}).$$

It is straightforward to see that, by Fubini's theorem,

$$\|f\|^p_{L^p(Q)} = \int_Q \int_0^{|f(x)|} p\alpha^{p-1} \, d\alpha dx = \int_0^\infty p\alpha^{p-1} \left(\int_{\{x \in Q: \ |f(x)| > \alpha > 0\}} dx \right) d\alpha = \int_0^\infty p\alpha^{p-1} \lambda_f(\alpha) \, d\alpha.$$

Then, for any cube Q,

$$\frac{1}{m(Q)}\int\limits_{Q}|f(x)-f_Q|^p\,dx = \frac{1}{m(Q)}\int\limits_{0}^{\infty}p\alpha^{p-1}\lambda_{|f-f_Q|^p}(\alpha)\,d\alpha \le c_1 p\int\limits_{0}^{\infty}\alpha^{p-1}e^{-c_2\alpha}\,d\alpha \le C_{n,p},$$

completing the proof. □

Next, we will briefly introduce Carleson measures, then apply them to BMO. Let $B = B(x_0, r)$ and define its *tent* $T(B) \subset \mathbb{R}^{n+1}_+$ as

$$T(B) = \{(x,t) \in \mathbb{R}^{n+1} :\ |x - x_0| \le r - t\}.$$

If μ is a Borel measure on $\mathbb{R}^{n+1}_+$, let $C(\mu)$ be the function

$$C(\mu)(x) = \sup_{B\in\mathcal{B}_x}\frac{1}{m(B)}\int\limits_{T(B)}|d\mu|,$$

where $\mathcal{B}_x$ is the collection of balls containing x. A measure for which $C(\mu)$ is bounded is called a *Carleson measure*, with *Carleson norm* $\|\mu\|_{\mathcal{C}} = \sup\limits_{x\in\mathbb{R}^n}|C(\mu)(x)|$.

First, we remark that for a BMO function f,

$$\int\limits_{\mathbb{R}^n}\frac{|f(x)|}{(1+|x|)^{n+\epsilon}}\,dx < \infty$$

for any $\epsilon > 0$, allowing us to define BMO as a subspace of tempered distributions $\mathcal{S}'$ (modulo constants), where $\mathcal{S}$ denotes the Schwartz space. This is important for the following reason:

Proposition 2.5. *Let $\Phi \in \mathcal{S}$, with $\int \Phi = 0$.*

(1) *If $f \in \mathrm{BMO}$, and $d\mu = |f * \Phi_t(x)|^2\ dxdt/t$, then $d\mu$ is a Carleson measure.*
(2) *If there exists $\Psi \in \mathcal{S}$ with $\int \Psi = 0$ so that*

$$\int \Phi_t * \Psi_t\ \frac{dt}{t} = \delta$$

(that is, Φ is non-degenerate),

$$\int\limits_{\mathbb{R}^n}\frac{|f(x)|}{1+|x|^{n+1}}\,dx < \infty,$$

and $d\mu = |f * \Phi_t(x)|^2\ dxdt/t$ is a Carleson measure, then $f \in \mathrm{BMO}$.

This gives another characterization of BMO, a Carleson measure version. This will be important for the analysis done in [7]. The proof is quite substantial, so we omit it (see [10]). As noted in [10], the hypotheses on Φ and Ψ can be reduced significantly. They discuss, for example, choosing $\Phi(x) = \frac{\partial}{\partial t}P_t(x)\big|_{t=1}$, where P_t is the Poisson kernel. Taking $\Psi = 4\Phi$ yields, as a consequence, that $f \in \mathrm{BMO}$ if and only if $t|\nabla(f * P_t)|^2\ dxdt$ is a Carleson measure.

3. The Space BMO^{-1}

First, let $\Phi(x) = \pi^{-n/2} e^{-|x|^2}$. Using a Carleson measure characterization of BMO, we say that $v \in \mathcal{S}'$ is in BMO if

$$\|v\|_{\mathrm{BMO}} = \sup_{x,r>0} \left(2m(B_r(x))^{-1} \int_{B_r(x)} \int_0^r t|\nabla(\Phi_t * v)|^2 \, dtdy \right)^{1/2} < \infty.$$

Recall that if w solves the heat equation with initial data v, then w is given by the convolution of v with the heat kernel, i.e. $w(t) = v * \Phi_{\sqrt{4t}}$ (provided sufficient assumptions on both v and w), which then allows us to re-write the norm as

$$\|v\|_{\mathrm{BMO}} = \sup_{x,r>0} \left(m(B_r(x))^{-1} \int_{B_r(x)} \int_0^{r^2} |\nabla(w)|^2 \, dtdy \right)^{1/2}.$$

We call w the *caloric extension* of v. Now, we define BMO^{-1} to be elements $v \in \mathcal{S}'$ for which

$$\|v\|_{\mathrm{BMO}^{-1}} := \sup_{x,r>0} \left(m(B_r(x))^{-1} \int_{B_r(x)} \int_0^{r^2} |w|^2 \, dtdy \right)^{1/2} < \infty.$$

A nice, evident property of BMO^{-1} is that quotienting by constants is not required to make it normed. We can characterize BMO^{-1} as follows:

Theorem 3.1. *Let $u \in \mathcal{S}'$. Then, $u \in \mathrm{BMO}^{-1}(\mathbb{R}^n)$ if and only if there exist $f^i \in \mathrm{BMO}(\mathbb{R}^n)$ so that $u = \sum_i \partial_i f^i$.*

That is, a tempered distribution is in BMO^{-1} if and only if it equals the divergence of a BMO vector field. The converse is trivial, but the forward direction is a bit more difficult. To prove that implication, we first provide a lemma.

Lemma 3.2. *Let $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be homogeneous of degree zero. Then,*

$$\|m(D_x)f\|_{\mathrm{BMO}^{-1}} \leq c \|f\|_{\mathrm{BMO}^{-1}}.$$

We delay the proof until after the proof of the theorem, which we can now show with relative ease.

Proof of 3.1. ($\Longleftarrow$) Suppose that $f^i \in \mathrm{BMO}$ for $1 \leq i \leq n$, and let v^i to be the caloric extension of each f^i. Further, let w be the caloric extension of u. Then, for any $x, r > 0$

$$m(B_r(x))^{-1} \int_{B_r(x)} \int_0^{r^2} |w|^2 \, dxdt \leq m(B_r(x))^{-1} \int_{B_r(x)} \int_0^{r^2} \sum_i |\partial_i v^i|^2 \, dxdt \leq \sum_i \|f^i\|^2_{\mathrm{BMO}} < \infty,$$

since each $f^i \in \mathrm{BMO}$. Thus, $u \in \mathrm{BMO}^{-1}$.

($\Longrightarrow$) Let $u \in \mathrm{BMO}^{-1}$. Call $R_{ij} = \partial_i \partial_j \nabla^{-1}$, and $u_{ij} = R_{ij} u$. Noting that if $m(D_x) = R_{ij}$, then the corresponding symbol is $m(\xi) = \frac{\xi_i \xi_j}{|\xi|^2}$. Hence, the lemma applies, yielding that

$$\|u_{ij}\|_{\mathrm{BMO}^{-1}} \leq c \|u\|_{\mathrm{BMO}^{-1}},$$

so $u_{ij} \in \mathrm{BMO}^{-1}$. Further, f^i exists so that $\partial_j f^i = u_{ij}$ (with $f^i = \partial_i \Delta^{-1} u$) which is well-defined by the fact that $\partial_k u_{ij} = \partial_i u_{kj}$. Each $f^i \in \mathrm{BMO}$ by construction, and further $u = \sum_i \partial_i f^i$, completing the proof. □

Now, we will provide the proof of the technical lemma.

Proof of 3.2. Let v denote the caloric extension of u. We first note that it will suffice to prove the result over a ball of radius 1 centered at 0 (BMO is both scale and translation invariant, which is easily seen from the standard definition, and so the same is true for BMO^{-1}). So, we must show that

$$\|m(D_x)v\|_{L^2(B_1(0)\times(0,1))} \leq c\,\|u\|_{\mathrm{BMO}^{-1}}.$$

To begin, we will demonstrate that

$$|v(x,t)| \leq ct^{-1/2}\,\|u\|_{\mathrm{BMO}^{-1}}.$$

First, we claim that scaling and translating reduces to the case of $x=0,\ t=1$. Indeed, the former comes from the fact that convolution is commutative and that BMO^{-1} is translation invariant, and the latter follows since changing variables to remove the $4t$ from the exponential will lead to cancellation from the $(4t)^{-n/2}$ factor in the heat kernel. That is, we must show that $|v(0,1)| \leq c\,\|u\|_{\mathrm{BMO}^{-1}}$. One can see without much difficulty that the left-hand side is bounded, since evaluation here makes the heat kernel a Schwartz function, and so the result follows immediately.

Next, call $T = m(D_x)$ and fix $t \in (0,1]$, and let $S(t)$ denote the heat semigroup. Then, if $t \in (0,1]$,

$$\begin{aligned} Tv(x,t) &= TS(t)u(x) = T(S(t)-S(1))u(x) + TS(1)u(x) \\ &= T(S(t)-S(1))u(x) + \int_1^\infty T\Delta S(s)u(x)\,ds \\ &= T(I - S(1-t))v(x,t)) + \int_1^\infty T\Delta S(s/2)v(x,s/2)\,ds, \end{aligned}$$

where we have used that $\partial_t S = \Delta S(t)$ and the fundamental theorem of calculus on the second line, as well as semigroup properties in the third (i.e. $S(s+t) = S(s)S(t)$ for $s,t \geq 0$). We will bound each term separately.

Now, since $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree 0, we have that T is bounded on L^2, and thus $T(I - S(1-t))$ is L^2-bounded. If we let $k(x,t) = T\left(\delta_0 - \Phi_{\sqrt{4(1-t)}}\right)$ be the kernel of the aforementioned operator (with fixed $t \in (0,1)$), then we have the kernel bound

$$|k_t(x)| \leq ct|x|^{-n-2} \leq c|x|^{-n-2}.$$

Using the aforementioned L^2 boundedness, Young's convolution inequality, and the kernel bound provides

$$\|T(I - S(1-t))v(t)\|_{L^2(B_1(0))} \leq c \sup_{x\in\mathbb{R}^n} \|v(\cdot,t)\|_{L^2(B_1(x))} \leq c\,\|u\|_{\mathrm{BMO}^{-1}},$$

for any $t \in (0,1)$. As for the second term, note that the kernel $k(x,t)$ of $T\Delta S(t)$ satisfies

$$\|k(\cdot,t)\|_{L^1(\mathbb{R}^n)} \leq ct^{-1}$$

for $t \in (0,1)$. Using this in together with Young's convolution inequality and the first bound that we demonstrated at the beginning of the lemma, we obtain

$$\left\| \int_1^\infty T\Delta S(s/2)v(\cdot,s/2)\,ds \right\|_{L^\infty(\mathbb{R}^n)} \le c\int_1^\infty s^{-1}\left\|v(\cdot,s/2)\right\|_{L^\infty(\mathbb{R}^n)}\,ds \le c\left\|u\right\|_{\mathrm{BMO}^{-1}}.$$

Note that this is an L^∞ bound, which is actually stronger than the needed L^2 bound. Putting the bounds on the two terms together yields the result. □

4. An Application of BMO^{-1} to the Incompressible Navier-Stokes Equations

The incompressible Navier Stokes equations are given on $\mathbb{R}^n \times \mathbb{R}^+$ by

$$\begin{cases} u_t + (u\cdot\nabla)u - \Delta u + \nabla p = 0 \\ \nabla\cdot u = 0 \\ u(0) = u_0, \end{cases}$$

where u represents velocity and p pressure. We implicitly assume an appropriate boundary condition on p at ∞. Global well-posedness for small initial data and local well-posedness for large initial data has been proven for initial data in $L^n(\mathbb{R}^n)$ (see [6]), various Morrey spaces $M_q^n(\mathbb{R}^n)$ (see [4] and [11]), and Besov spaces $B_{p,\infty}^{-1+n/p}(\mathbb{R}^n)$, with $1 < p < \infty$ and $p > n$ (see [3] and [9]). An open problem is to find the largest space (with respect to initial data) for local or global solutions exist. In [7], Koch and Tataru consider BMO^{-1} as a viable candidate.

First, if we let $Q(x,r) = B_r(x)\times(0,r^2)$, then they define normed spaces X and Y via the quasi-mixed norms

$$\|u\|_X = \sup_t t^{1/2}\left\|u(\cdot,t)\right\|_{L^\infty(\mathbb{R}^n)} + \left(\sup_{x,r>0} m(B_r(x))^{-1}\int_{Q(x,r)}|u|^2\,dydt\right)^{1/2}$$

and

$$\|f\|_Y = \sup_t t\left\|f(\cdot,t)\right\|_{L^\infty(\mathbb{R}^n)} + \sup_{x,r>0} m(B_r(x))^{-1}\int_{Q(x,r)}|f|\,dydt.$$

They also define local spaces BMO_r in a manner analogous to BMO, but where the radii of the balls in the sup are, at most, r (and similarly for other spaces, such as X, Y, BMO^{-1}, etc.). Now, if $v \in \mathrm{BMO}_1$, we say that $v \in \overline{\mathrm{VMO}}$ if and only if $\|v\|_{\mathrm{BMO}_r} \to 0$ as $r \to 0$, and similarly for $\overline{\mathrm{VMO}}^{-1}$ (recall VMO as the space of vanishing mean oscillation, defined as the closure of BMO in continuous functions that vanish at infinity, or where the mean oscillation goes to 0 uniformly as the radius goes to 0 or infinity). Now, we can state the theorems of Koch and Tataru.

Theorem 4.1 (Global Version). *The incompressible Navier-Stokes equations have a unique, small, global solution in X for all initial data $u_0 \in \mathrm{BMO}^{-1}$ for which $\nabla\cdot u_0 = 0$, with sufficiently small BMO^{-1} norm.*

Theorem 4.2 (Local Version). *There exists $\epsilon > 0$ so that for any $r > 0$, the incompressible Navier-Stokes equations have a unique, small solution in X_r (up to $t = r^2$), provided that $\nabla\cdot u_0 = 0$ and $\|u_0\|_{\mathrm{BMO}_r^{-1}} \le \epsilon$. In particular, for all divergence-free $u_0 \in \overline{\mathrm{VMO}}^{-1}$, there exists a unique, small, local solution.*

We will provide a general outline of the proof (of the global version). First, write the Navier-Stokes equations in integral form, using the Leray projection and the inhomogeneous heat equation parametrix. In order to apply a Picard contraction-type theorem, one must show that various maps send the right spaces to the right spaces. In particular, one must show that $V\nabla\Pi N(u)$ is a map from X into X, where V is the inhomogeneous heat equation parametrix, Π is the Leray projection, and $N(u) = u \otimes u$. This reduces to showing that $V\nabla\Pi$ sends Y into X. So, one needs a pointwise estimate and an L^2 estimate. First, the do a case reduction to looking at more simple regions, then they localize in these simple regions. The pointwise estimate is proved via an estimate on the kernel of $V\nabla\Pi$. The L^2 estimate is quite complicated. A remark outlines another proof, where it suffices to prove another pointwise estimate, as well as a different L^2 estimate. The pointwise estimate is obvious after scaling, and the L^2 estimate is obtained by using the typical energy inequality.

Subsequently, they show that $L^n(\mathbb{R}^n) \subset B^{-1+n/p}_{p,\infty}(\mathbb{R}^n) \subset \mathrm{BMO}^{-1}$ and $M^n_q \subset \mathrm{BMO}^{-1}_1$ for $1 < q \leq n$, so BMO^{-1} (to date) provides the most general function spaces for which divergence-free initial data can live to obtain solution existence. To further underscore the importance of the BMO^{-1} space, it is thought to be the largest space that possesses the scaling of $L^n(\mathbb{R}^n)$ where the incompressible Navier Stokes equations are well-posed ([1]). In fact, it has been shown that the problem is ill-posed in the largest possible space $B^{-1}_{\infty,\infty}(\mathbb{R}^n)$ in [2] and in a space between the two in [12].

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