Asymptotic Stability of Biharmonic Shallow Water Equations

by

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Abstract

The dissipative shallow-water equations (SWE) possess both real-world application and extensive analysis in theoretical partial differential equations. This analysis is dominated by modeling the dissipation as diffusion, with its mathematical representation being the Laplacian. However, the usage of the biharmonic as a dissipative operator by oceanographers and atmospheric scientists and its underwhelming amount of analysis indicates a gap in SWE theory. In order to provide rigorous mathematical justification for the utilization of these equations in simulations with real-world implications, we extend an energy method utilized by Matsumura and Nishida for initial value problems relating to the equations of motion for compressible, vsicous, heat-conductive fluids ([6], [7]) and applied by Kloeden to the diffusive SWE ([4]) to prove global time existence of classical solutions to the biharmonic SWE. In particular, we develop appropriate a priori growth estimates that allow one to extend the solution's temporal existence infinitely under sufficient constraints on initial data and external forcing, resulting in convergence to steady-state.

Contents

List of Symbols

Symbols

$\overline{f}$ The spatial average of an integrable function f over Ω is given by

$$\overline{f} = \frac{1}{|\Omega|}\int_{\Omega} f(x)dx.$$

$L^2(\Omega)$ This denotes the space of measurable functions that are square integrable over Ω, equipped with the norm

$$\|f\|_{L^2(\Omega)} = \left(\int_{\Omega} f(x)^2 \ dx\right)^{\frac{1}{2}}.$$

$H^k(\Omega)$ This denotes the Sobolev space of functions defined over Ω with k weak derivatives belonging to $L^2(\Omega)$. This space is equipped with the norm

$$\|f\|_k = \left(\sum_{l=0}^{k} \left\|D^l f\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}},$$

where $D^l f$ represents the lth generalized derivative of f. We utilize the notation $H^k(\Omega)$, instead of $W^{k,2}(\Omega)$, since this function space is a Hilbert Space.

$C^j(0,T;H^k(\Omega))$ This denotes the function space with elements $f : [0,T] \to H^k(\Omega)$ whom are j times continuously differentiable in t.

$L^2(0,T;H^k(\Omega))$ This denotes the function space of measurable functions

$f : [0,T] \to H^k(\Omega)$ whose norm $\|f(t)\|_k$ is square integrable over $[0,T]$.

$C^{k,\alpha}(\overline{\Omega})$ This space represents the space of functions that are k times continuously differentiable with the kth partial derivatives being Hölder continuous over $\overline{\Omega}$ (the closure of Ω) with Hölder coefficient $0 < \alpha \leqslant 1$. In this paper, we will require that $0 < \alpha < 1$ (this is dictated by the Sobolev emeddings present in Theorem 5.4 in [1]). If the k is dropped from the notation, it is understood that $k = 0$. This space is equipped with the norm

$$\|f\|_{C^{k,\alpha}(\overline{\Omega})} = \sum_{l=0}^{k} \left\|D^l f\right\|_{C^\alpha(\overline{\Omega})},$$

where

$$\left\|D^l f\right\|_{C^\alpha(\overline{\Omega})} = \sup_{x\in\Omega} |f(x)| + \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|D^l f(x) - D^l f(y)|}{|x-y|^\alpha}.$$

Note When our functions take values in $\mathbb{R}^2$, then interpret the L^2 norm as

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f(x)|^2 \, dx\right)^{\frac{1}{2}},$$

where $|\cdot|$ denotes the Euclidean 2-norm. All of the other spaces are adapted similarly.

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1

Introduction

1.1 Background

The Shallow Water Equations (SWE) constitute a prevalent system of hyperbolic/parabolic, non-linear partial differential equations in fluid dynamics. The SWE are derived from the conservation of mass and linear momentum, in the case where the velocity of the flow varies negligibly in the vertical direction, allowing for a hydrostatic approximation. Since scenarios where the horizontal length scale greatly exceeds the vertical length scale arise commonly in nature (particularly in the atmosphere and ocean), these equations possess practical usage in the atmospheric science and oceanography communities.

In order to mathematically characterize the SWE from a realistic perspective, first let $\Omega \subseteq \mathbb{R}^2$ be a rectangular domain. Define the fluid flow over Ω by $\mathbf{u} : \Omega \times [0, +\infty) \to \mathbb{R}^2$ and the height field $h : \Omega \times [0, +\infty) \to \mathbb{R}$. If we associate x with the east-west direction and y with the north-south direction, then the shallow water equations can be given, as in Jones ([3]), by

$$\begin{cases} h_t + \nabla \cdot (h\mathbf{u}) = 0 \\ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu D_L \mathbf{u} = -g' \nabla h - f(\hat{z} \times \mathbf{u}) + \mathbf{F}. \end{cases}$$

Here, f represents the Coriolis force (which we take to vary linearly with latitude),

g' is reduced gravity, ν is a viscosity parameter, D_L is a dissipative operator (we understand differential operators acting on $\mathbf{u}$ component-wise), $\mathbf{F}$ is external forcing, and $\hat{z}$ is a unit vector in the vertical direction. The external forcing takes the form $\mathbf{F} = (-\tau \cos(\pi \mathbf{u}_1/L_y), 0)$, where L_y is the north-south basin size and τ is wind stress. Take $\mathbf{u} = \mathbf{0}$ initially, with Dirichlet boundaries. (We will take various simplifications of these equations in subsequent sections of this paper.)

1.2 Biharmonic Vs. Laplacian

As mentioned previously, the SWE (particularly of the above variety) bear substantial geophysical relevance. The most important application of these equations are numerical simulations. One of the most common representations of the dissipation is diffusion, given by the Laplacian, $\nabla(h\nabla\mathbf{u})$. There are also multiple other relevant operators, such as the biharmonic operator, which we will take to be given by $\nabla^2(h\nabla\mathbf{u})$. The biharmonic is beneficial in numerical simulations, since it "smooths" differently than the Laplacian. Another viable way to modify the SWE is through a dispersive modification, via the use of the α-model (see [10]).

If one considers the standard one-dimensional heat equation with the Laplace versus biharmonic operators, both equipped with Dirichlet boundaries, the spectrum of the biharmonic operator is proportional to n^4, whereas the Laplacian's is proportional to n^2. This means that the biharmonic version will dissipate slower than the Laplacian at first, then much faster for larger n. From a numerical perspective, the biharmonic's spectrum can be viewed as indicating that it neglects to damp solutions until the wave numbers becomes large. This is highly beneficial, since the biharmonic refrains from artificially damping solutions at small scales and heavily damps the large wave numbers that threaten to blow up the simulation, with the large-scale dynamics already being resolved by the grid (see [2],[5],[9]). This makes it ideal for scenarios such as large-scale eddy simulations.

On the following page, we have a side-by-side comparison numerical solutions of the two operators for the previously-described version of the SWE, with Earth-like parameters. The fluid moves tangentially to the height contours, which emphasizes how it is influenced by the gyres that are generated from the $\beta-$plane approximation.

Note how the biharmonic produces a much "sharper" flow, which underscores the aforementioned smoothing.

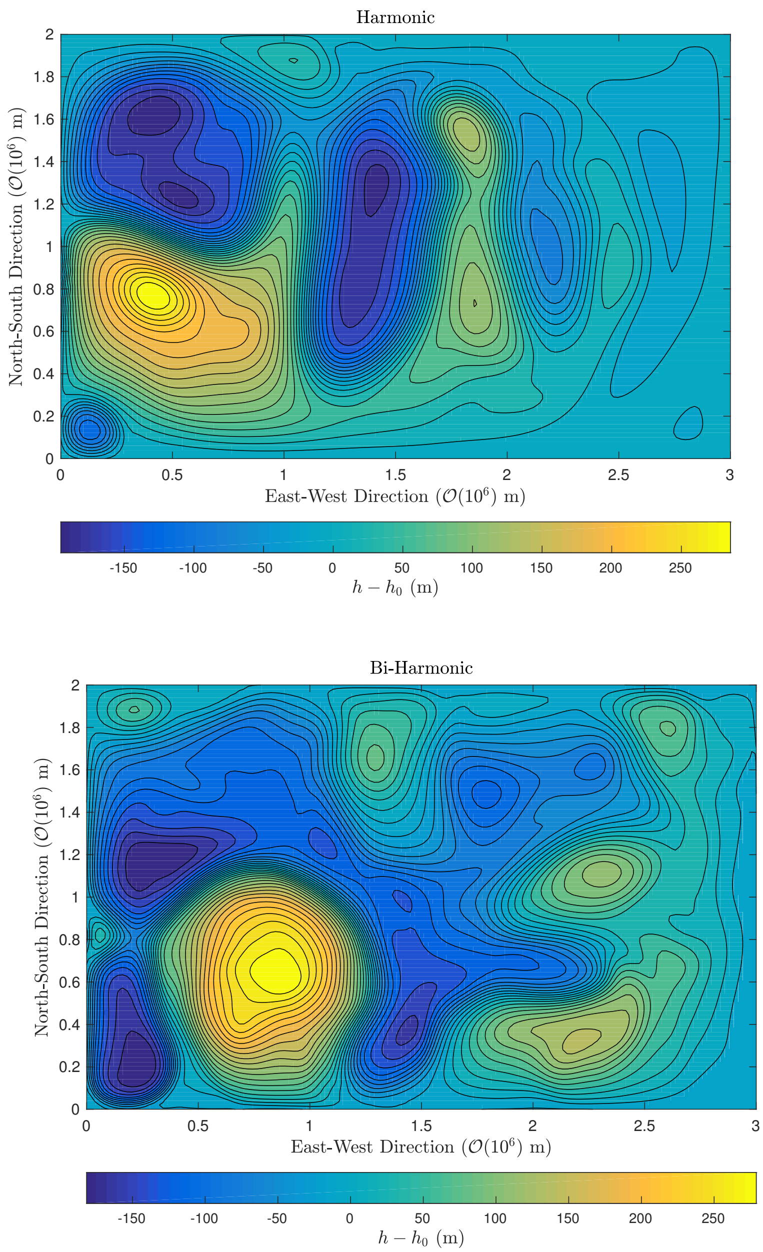

FIGURE 1.1: Height Field Contours of Diffusive and Biharmonic SWE on $\beta-$plane; Scheme: MPDATA

In short, while the Laplacian is a common dissipative operator among oceanographers and atmospheric scientists, there are numerous other operators with legitimate numerical bases, such as the biharmonic. However, theoretical analysis of these equations, which help provide justification for the aforementioned numerical simulations, is dominated by the Laplacian. There is a sizable gap in the theory about alternative operators. In this paper, we aim to establish justification for the usage of the biharmonic shallow water equations in numerical simulations under initial data and forcing contraints by proving the global existence of classical solutions. This has been done previously by P.E. Kloeden (see [4]) for the Laplacian, via an energy method developed by A. Matsumura and T. Nishida (see [6], [7]) for more general fluids. We will follow a similar approach to [4], avoiding the complicated boundary estimates present in [6], [7].

2

Mathematical Setting

As mentioned previously, we will make various modifications to the previously-described SWE. One such modification is that we will take $\Omega = (0,1) \times (0,1)$, Another is that we will consider doubly-periodic flows, allowing us to ascribe periodic boundaries to our system. These conditions establish integration by parts as a viable strategy when performing energy estimates. Finally, while the biharmonic operator is often taken to be $\nabla^2(h\nabla^2\mathbf{u})$, we will take it to be $\nabla^4\mathbf{u}$, which we will discuss later. By non-dimensionalizing and taking the forcing $\boldsymbol{\phi}(\mathbf{x})$ as being produced by a potential $V : \Omega \to \mathbb{R}$ given by $-\nabla V(\mathbf{x}) = \boldsymbol{\phi}(\mathbf{x})$, we obtain the following brand of the SWE:

$$\begin{cases} h_t + \nabla \cdot (h\mathbf{u}) = 0 \\ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \lambda h^{-1}\nabla^4\mathbf{u} + \nabla h = \boldsymbol{\phi} \\ \overline{\mathbf{u}} = \mathbf{0} \\ h(\mathbf{x}, t) > 0 \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \\ h(\mathbf{x}, 0) = h_0(\mathbf{x}), \end{cases} \tag{2.1}$$

all over Ω, along with the previously-mentioned periodic boundary conditions (let these conditions be contained in the label (2.1), as well). Here, $\lambda \in \mathbb{R}^+$ is the viscosity parameter associated with the biharmonic operator, and $h^{-1} := \frac{1}{h}$ (this is a slight abuse of notation used for convenience). The additional conditions present in the above are a positivity constraint on the height and a zero-average condition on the velocity (again, both over Ω). The steady-state solutions $\hat{\mathbf{u}}(\mathbf{x})$ and $\hat{h}(\mathbf{x})$ of (2.1) solve the system

$$\begin{cases} \nabla \cdot (\hat{h}\hat{\mathbf{u}}) = 0 \\ (\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}} + \lambda \hat{h}^{-1}\nabla^4 \hat{\mathbf{u}} + \nabla \hat{h} = \boldsymbol{\phi} \end{cases} \tag{2.2}$$

with analogous conditions as in (2.1) (again, let the conditions be contained in the label (2.2)).

Remark 2.1. One can see that choosing the forcing to be produced by a potential allows us to obtain a trivial steady-state for $\mathbf{u}$. Generalizing the forcing obfuscates the uniqueness of the steady-state equations significantly. △

Remark 2.2. One can show that the positivity constraint carries over from (2.1) to (2.2), under constraints on the forcing. Indeed, let $\overline{h}_0 \in \mathbb{R}^+$ and V be an arbitrary potential. Define $\hat{h}(\mathbf{x}) = \overline{h}_0 + \overline{V} - V(\mathbf{x})$. By an application of the embedding of $H^2(\Omega)$ into $C^\alpha(\overline{\Omega})$, given by [1] (Theorem 5.4) and the Poincaré Inequality (see [11]), we see that

$$\begin{aligned} |V(\mathbf{x}) - \overline{V}| &\leqslant \sup_{\mathbf{x}\in\Omega}\{|V(\mathbf{x}) - \overline{V}|\} \leqslant C_1 \left\| V(\mathbf{x}) - \overline{V} \right\|_2 \\ &\leqslant C_1 C_2 \left\| DV \right\|_1 = C \left\| DV \right\|_1, \end{aligned}$$

where C_1 depends only Ω, C_2 depends only on Ω and p in the L^p-space in question (for us, $p = 2$), and $C = C_1 C_2$. Suppose that

$$\left\| DV \right\|_5 \leqslant \frac{\overline{h}_0}{4C} = E_1(\overline{h}_0). \tag{2.3}$$

Then, in particular,

$$|V(\mathbf{x}) - \overline{V}| \leqslant \frac{\overline{h}_0}{4},$$

which implies that

$$|\hat{h}(\mathbf{x}) - \overline{h}_0| \leqslant \frac{\overline{h}_0}{4},$$

or, by the triangle inequality,

$$0 < \frac{3\overline{h}_0}{4} \leqslant \hat{h}(\mathbf{x}),$$

for all $\mathbf{x} \in \Omega$, which shows that $\hat{h}(\mathbf{x})$ is positive over Ω.

From here, we can see that, under the above constraints, a pair $(\hat{h}, \hat{\mathbf{u}})$ of the form $(\overline{h}_0 + \overline{V} - V(\mathbf{x}), 0)$ solves (2.2), where $\overline{\hat{h}} = \overline{h}_0$. We will show uniqueness in the subsequent section. △

Our main result that we shall prove is the following theorem.

Theorem 2.1. *(Global Solution Existence) Let h_0, $\mathbf{u}_0$, and V be doubly-periodic functions, where $h_0 \in H^5(\Omega)$, $(\mathbf{u}_0)_j \in H^6(\Omega)$ for $j = 1, 2$, $V \in H^6(\Omega)$, and $h_0 > 0$ over Ω. Then, $\delta > 0$ exists so that if*

$$\left\|h_0 - \overline{h}_0\right\|_5 \leqslant \delta, \quad \|\mathbf{u}_0\|_6 \leqslant \delta, \quad \|DV\|_5 \leqslant \delta,$$

then (2.1) and (2.2) have unique solutions, and

$$\lim_{t\to\infty} \sup_{\mathbf{x}\in\Omega} \{|h(\mathbf{x}, t) - \hat{h}(\mathbf{x}, t)|, |\mathbf{u}(\mathbf{x}, t) - \hat{\mathbf{u}}(\mathbf{x})|\} = 0, \tag{2.4}$$

where

$$\begin{aligned} &\hat{h}(\mathbf{x}) = \overline{h}_0 + \overline{V} - V(\mathbf{x}) \\ &\hat{\mathbf{u}}(\mathbf{x}) = \mathbf{0} \\ &h - \hat{h} \in C^0(0, +\infty; H^5(\Omega)) \cap C^1(0, +\infty; H^5(\Omega)) \\ &\mathbf{u}_j \in C^0(0, +\infty; H^6(\Omega)) \cap C^1(0, +\infty; H^2(\Omega)), \ j = 1, 2. \end{aligned} \tag{2.5}$$

Remark 2.3. This δ can be controlled to be as small as necessary by shrinking the initial data and external forcing. △

The solutions to both (2.1) and (2.2) are understood to be classical solutions and continous over $\Omega \times [0, +\infty)$, which can be seen from the embedding of $H^{k+2}(\Omega)$ into $C^{k,\alpha}(\overline{\Omega})$, (see [1], Theorem 5.4).

The theorem states the intuitive and deceptively-complicated result that taking initial data and forcing small enough causes the solution to converge to its steady-state. As we will see, however, the proof is non-trivial.

3

Proof of the Main Theorem

The general method of proof is as follows:

1. Prove the uniqueness of steady-state.

2. Perturb our solution about the steady-state.

3. Obtain local existence of the perturbation equation solutions.

4. Prove that we can bound the energy norm (to be discussed shortly) over some interval by a multiple of the initial energy norm.

5. Prove that we can iteratively extend the interval of existence on our solution by continually administering the a priori estimate given by our local bounds on the energy norm.

6. Obtain the asymptotic properties given by (2.4), completing the proof.

Following this outline, we proceed to the first proposition. It is readily seen that the pair $(\hat{h}, \hat{\mathbf{u}})$ given in Theorem (2.1) solves (2.2). It remains to show that this representation is unique.

Proposition 3.1 (Uniqueness of Steady-State). *Let* $\overline{h}_0 \in \mathbb{R}^+$ *be arbitrary and* $V(\mathbf{x}) \in H^6(\Omega)$ *be a doubly-periodic function satisfying* (2.3). *Then* $(\hat{h}(\mathbf{x}), \hat{\mathbf{u}}(\mathbf{x}))$, *as defined in Theorem* (2.1), *are the unique solution to* (2.2), *where* $\hat{\mathbf{u}}_j \in H^4(\Omega)$ *for* $j = 1, 2$.

The standard method of proof for uniqueness results, after obtaining existence, involves supposing that there are two solutions, taking their difference, obtaining a new system of PDEs, then showing that the difference is zero. The non-linearity makes this process exceedingly messy and complicated. Instead, we show that the solution exists, and that it *must* take one particular form.

Proof of Proposition (3.1). Fix $\overline{h}_0$, and suppose that $V \in H^6(\Omega)$ is defined as in (2.3). Multiplying the momentum equation in (2.2) by $\hat{h}\hat{\mathbf{u}}$, then integrating over Ω, we have

$$\int_\Omega (\hat{h}\hat{\mathbf{u}}) \cdot ((\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}})\ d\mathbf{x} + \lambda \int_\Omega \hat{\mathbf{u}} \cdot \nabla^4 \hat{\mathbf{u}}\ d\mathbf{x} + \int_\Omega (\hat{h}\hat{\mathbf{u}}) \cdot \nabla \hat{h}\ d\mathbf{x} = \int_\Omega (\hat{h}\hat{\mathbf{u}}) \cdot \boldsymbol{\phi}\ d\mathbf{x}.$$

Recalling that that $-\nabla V = \boldsymbol{\phi}$, we can re-write the above as

$$\int_\Omega (\hat{h}\hat{\mathbf{u}}) \cdot ((\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}})\ d\mathbf{x} + \lambda \int_\Omega \hat{\mathbf{u}} \cdot \nabla^4 \hat{\mathbf{u}}\ d\mathbf{x} + \int_\Omega (\hat{h}\hat{\mathbf{u}}) \cdot \nabla (\hat{h} + V)\ d\mathbf{x} = 0.$$

We will analyze these integrals individually. Recalling that differential operators act on $\hat{\mathbf{u}}$ component-wise and integrating the first by parts, we obtain that

$$\begin{aligned}\int_\Omega (\hat{h}\hat{\mathbf{u}}) \cdot ((\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}})\ d\mathbf{x} &= \frac{1}{2}\int_\Omega (\hat{h}\hat{\mathbf{u}}) \cdot \nabla(\hat{\mathbf{u}} \cdot \hat{\mathbf{u}})\ d\mathbf{x} \\ &= \frac{1}{2}\left(\oint_{\partial\Omega} (\hat{\mathbf{u}} \cdot \hat{\mathbf{u}})\hat{h}\hat{\mathbf{u}} \cdot \mathbf{n}\ dx - \int_\Omega \nabla \cdot (\hat{h}\hat{\mathbf{u}})(\hat{\mathbf{u}} \cdot \hat{\mathbf{u}})\ d\mathbf{x} \right),\end{aligned}$$

where $\mathbf{n}$ denotes the outward unit normal on Ω. The first term vanishes by the periodic boundary conditions, and the second term is 0, since $\nabla \cdot (\hat{h}\hat{\mathbf{u}}) = 0$.
Next, consider the second integral. Integrating by parts again, we have

$$\begin{aligned}&\lambda \int_\Omega \hat{\mathbf{u}} \cdot \nabla^4 \hat{\mathbf{u}}\ d\mathbf{x} \\ &= \lambda \left(\sum_{j=1}^{2} \oint_{\partial\Omega} (\hat{\mathbf{u}}_j \nabla^3 \hat{\mathbf{u}}_j) \cdot \mathbf{n}\ dx - \sum_{j=1}^{2} \oint_{\partial\Omega} (\nabla^2 \hat{\mathbf{u}}_j \nabla \hat{\mathbf{u}}_j) \cdot \mathbf{n}\ dx + \int_\Omega (\nabla^2 \hat{\mathbf{u}} \cdot \nabla^2 \hat{\mathbf{u}})\ d\mathbf{x} \right).\end{aligned}$$

The first two terms goes to 0 from the given boundary conditions. We will discuss the third piece in a moment. Shifting our attention to the third integral temporarily, we will integrate by parts, again:

$$\int_\Omega (\hat{h}\hat{\mathbf{u}}) \cdot \nabla(\hat{h} + V)\ d\mathbf{x} = \oint_{\partial\Omega} (\hat{h} + V)\hat{h}\hat{\mathbf{u}} \cdot \mathbf{n}\ dx - \int_\Omega \nabla \cdot (\hat{h}\hat{\mathbf{u}})(\hat{h} + V)\ d\mathbf{x}.$$

For the same reasons as the first integral, this integral is 0. Combining these results,

$$\lambda \int_\Omega (\nabla^2 \hat{\mathbf{u}} \cdot \nabla^2 \hat{\mathbf{u}})\ d\mathbf{x} = 0.$$

Since $\lambda \neq 0$, it follows that $\nabla^2 \hat{\mathbf{u}} = \mathbf{0}$. Multiplying both sides by $\hat{\mathbf{u}}$ and integrating over the domain,

$$\int_\Omega \hat{\mathbf{u}} \cdot \nabla^2 \hat{\mathbf{u}}\ d\mathbf{x} = 0.$$

Integrating by parts,

$$\int_\Omega \hat{\mathbf{u}} \cdot \nabla^2 \hat{\mathbf{u}}\ d\mathbf{x} = \sum_{j=1}^{2} \oint_{\partial\Omega} (\hat{\mathbf{u}}_j \nabla \hat{\mathbf{u}}_j) \cdot \mathbf{n})\ dx - \|\nabla \hat{\mathbf{u}}\|^2_{L^2(\Omega)} = 0.$$

The first term is 0 from the boundary conditions. The second term implies that $\nabla \hat{\mathbf{u}} = 0$. Hence, $\hat{\mathbf{u}}$ is constant on Ω. But, we are given that $\hat{\mathbf{u}}$ has zero average on Ω, and $\hat{\mathbf{u}}$ is continuous. Then, $\hat{\mathbf{u}} = 0$. Applying this to the momentum equation yields that

$$\nabla(\hat{h} + V) = \mathbf{0}.$$

So, $\hat{h} + V$ is constant, from which we can conclude that $\hat{h} + V = \overline{h}_0 + \overline{V}$ is unique. □

Next, we wish to perturb $(h, \mathbf{u})$ about the steady-state solution $(\hat{h}, \hat{\mathbf{u}})$. Define the perturbation variables $(h', \mathbf{u}') = (h - \hat{h}, \mathbf{u} - \hat{\mathbf{u}}) = (h - \hat{h}, \mathbf{u})$. Re-arranging and plugging these variables into (2.1), then dropping primes, yields the perturbation equations:

$$\begin{aligned} h_t + \mathbf{u} \cdot \nabla h + \overline{h}_0 \nabla \cdot \mathbf{u} &= R_0 \\ \mathbf{u}_t + \lambda \nabla^4 \mathbf{u} + \nabla h &= \mathbf{R}, \end{aligned} \tag{3.1}$$

where

$$R_0 = (\overline{h}_0 - h - \hat{h})\nabla \cdot \mathbf{u} - \nabla\hat{h} \cdot \mathbf{u}$$
$$\mathbf{R} = -(\mathbf{u} \cdot \nabla)\mathbf{u},$$

with analogous boundary conditions to (2.1) and (2.2) (note that the zero-average condition carries over, since $\mathbf{u}' = \mathbf{u}$), as well as an identical initial condition on $\mathbf{u}$ and the initial condition $h(\mathbf{x}, 0) = h_0(\mathbf{x}) - \hat{h}(\mathbf{x})$ over Ω. We will refer to the left-hand sides of (3.1) as L_0 and $\mathbf{L}$, respectively.

Remark 3.1. If we did not simplify the operator, the perturbation equations would be given by

$$\begin{aligned} h_t + \mathbf{u} \cdot \nabla h + \overline{h}_0 \nabla \cdot \mathbf{u} &= R_0 \\ \mathbf{u}_t + \lambda \nabla^4 \mathbf{u} + \nabla h &= \mathbf{R}, \end{aligned} \tag{3.2}$$

where

$$R_0 = (\overline{h}_0 - h - \hat{h})\nabla \cdot \mathbf{u} - \nabla\hat{h} \cdot \mathbf{u}$$
$$\mathbf{R} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - 2\lambda \frac{\nabla(h + \hat{h})}{(h + \hat{h})} \nabla^3 \mathbf{u} - \frac{\lambda}{(h + \hat{h})} \nabla^2 (h + \hat{h}) \nabla^2 \mathbf{u},$$

with analogous boundary conditions to (2.1) and (2.2), as well as identical initial condition on $\mathbf{u}$ and the initial condition $h(\mathbf{x}, 0) = h_0(\mathbf{x}) - \hat{h}(\mathbf{x})$ over Ω. Since the first equation is identical, many estimates follow identically to those that will follow. The differences are the bounds on $\mathbf{R}$, with the primary issue being reducing the order of differentiation sufficiently. We will attach a key estimate on the term $h + \hat{h}$ in the appendix, as well as an example estimate for $\mathbf{R}$, for future work. △

The perturbation equation functions live in the function space

$$\begin{aligned}\mathscr{X} &= \mathscr{X}(t_1, t_2; E)\\ &= \Big\{(h, \mathbf{u}); h \in C^0(t_1, t_2; H^5(\Omega)) \cap L^2(t_1, t_2; H^5(\Omega)),\\ &\qquad h_t \in C^0(t_1, t_2; H^5(\Omega)) \cap L^2(t_1, t_2; H^5(\Omega)),\\ &\qquad \mathbf{u}_j \in C^0(t_1, t_2; H^6(\Omega)) \cap L^2(t_1, t_2; H^8(\Omega)),\\ &\qquad (\mathbf{u}_t)_j \in C^0(t_1, t_2; H^2(\Omega)) \cap L^2(t_1, t_2; H^4(\Omega)),\ j = 1, 2\Big\},\end{aligned} \tag{3.3}$$

where $N(h, u; t_1, t_2) \leqslant E$. The quantity $N(h, u; t_1, t_2)$ is called the *energy norm* of (3.1), and it is defined as

$$\begin{aligned}N^2(h, \mathbf{u}; t_1, t_2) = \sup_{t_1 \leqslant \tau \leqslant t_2} &\Big\{ \|h(\tau)\|_5^2 + \|h_\tau(\tau)\|_5^2 + \|\mathbf{u}(\tau)\|_6^2 + \|\mathbf{u}_t(\tau)\|_2^2 \Big\}\\ &+ \int_{t_1}^{t_2} \left(\|Dh(\tau)\|_4^2 + \|h_\tau(\tau)\|_5^2 + \|D\mathbf{u}(\tau)\|_7^2 + \|\mathbf{u}_\tau(\tau)\|_4^2\right) d\tau.\end{aligned} \tag{3.4}$$

We will denote it by $N^2(t_1, t_2)$. This quantity is paramount to proving our result, since it defines when the norms of all relevant quantities are still bounded.

Remark 3.2. Given the solutions $\hat{h}$ from (2.2) and h from (3.1), under the condition (2.3), we can define a bound on the energy norm that admits positivity on the original height and the original height minus its domain average. Note that, by using the embedding of $H^2(\Omega)$ into $C^\alpha(\overline{\Omega})$,

$$|h(\mathbf{x}, t)| \leqslant \sup_{\mathbf{x} \in \Omega} |h(\mathbf{x}, t)| \leqslant C \|h(t)\|_2 \leqslant C \sup_{t \in [0,T]} \|h(t)\|_2 \leqslant CN(0, T),$$

where C is a time-independent constant coming from the embedding. Suppose that the energy norm is bounded over $[0, T]$ by $\dfrac{\overline{h}_0}{2C}$ (for justification of this supposition, see [7]). That is,

$$N(0, T) \leqslant \frac{\overline{h}_0}{2C} = E_2(\overline{h}_0). \tag{3.5}$$

Then,

$$|h(\mathbf{x},t)| \leqslant \frac{\overline{h}_0}{2},$$

from which we can conclude from (2.3) that

$$|h(\mathbf{x},t) + \hat{h} - \overline{h}_0| \leqslant |h(\mathbf{x},t)| + |\hat{h} - \overline{h}_0| \leqslant \frac{\overline{h}_0}{2} + \frac{\overline{h}_0}{4} = \frac{3\overline{h}_0}{4},$$

and

$$h(\mathbf{x},t) + \hat{h}(\mathbf{x}) \geqslant \frac{\overline{h}_0}{4} > 0. \tag{3.6}$$

△

With this in mind, we move onto the subsequent requirement for our proof.

Proposition 3.2 (Local Existence)**.** *Suppose that* $\|DV\|_5 \leqslant E_1(\overline{h}_0)$ *and that* (3.1) *with the given conditions have a doubly-periodic solution* $(h, \mathbf{u})$ *on* $[0, T]$ *such that* $N(0,T) \leqslant E_2(\overline{h}_0)$ *for some* $T \geqslant 0$*. Then, there exist real, positive,* T*-independent constants* $\tau, \epsilon_0, C_0 > 0$*, with* $\epsilon_0\sqrt{1 + C_0^2} \leqslant E_2(\overline{h}_0)$*, so that if* $N(T,T) \leqslant \epsilon_0$*, then the perturbation equations have a unique solution on* $[T, T+\tau]$ *such that*

$$N(T, T+\tau) \leqslant C_0 N(T,T).$$

Remark 3.3. In a sense, the above proposition is a conjecture, as it will not be proven in this paper. However, literature indicates that the result should hold, as it holds for operators such as the Laplacian. The local existence for the SWE becomes problematic at scales where the biharmonic damps solutions more than the Laplacian, so local existence of the Laplacian should signify the analgous result for the biharmonic. The standard method of proof for a result such as this is an argument utilizing successive approximations to construct a sequence that is shown to be Cauchy in an appropriate Banach Space. △

Next, we obtain a key a priori estimate on the growth of the energy norm.

Proposition 3.3 (A Priori Growth Estimate)**.** *Suppose that* (3.1) *with the given conditions have a doubly-periodic solution* $(h, \mathbf{u})$ *over* $[0, T]$ *so that* $N(0, T) \leqslant E_2(\overline{h}_0)$ *for some* $T > 0$*. Then, there exist positive, real,* T*-independent constants* ϵ_1 *and* C_1*, with* $\epsilon_1 < \epsilon_0$ *and* $\epsilon_1 C_1 \leqslant E_2(\overline{h}_0)$*, so that if* $N(0, T) \leqslant \epsilon_1$ *and* $\|DV\|_5 \leqslant \epsilon_1$*, then*

$$N(0, T) \leqslant C_1 N(0, 0).$$

As before, we can make ϵ_1 as small as necessary by shrinking the initial data.

Remark 3.4. The local existence both provides a local solution and a means of extending the interval of existence of the solution iteratively by the re-initialized energy norm. In contrast, the a priori growth estimate provides a bound on norm growth over an entire interval by only the initial data. These come together in a natural way.

Here is a general overview of the process. Recall that the left-hand sides of (3.1) are bilinear. We can treat this as an iterative system, where the previous iterate comes from the right-hand side, allowing us to solve for the left. The initial iterate is in terms of the initial data and initial energy norm. On the first iteration, quantities in the system are bounded by our a priori estimates, which are then bounded by the initial energy norm and the forcing, which are given to be small by the local solution. On the next iteration, we obtain similar bounds, but these terms are themselves in terms of the first iteration, namely the initial data. This process continues, and it works based on the given local existence and sufficient a priori growth estimates. For details on the argument, see [7]. $\triangle$

Given a local solution, we can iteratively extend the interval of existence for our solution globally, when provided with the above a priori estimate. This proposition is the crux of our argument, and it will requires numerous estimates on terms present in the energy norm. In the first lemma, we obtain a Poincaré Inequality on the height (it is given immediately on $\mathbf{u}$, since $\mathbf{u}$ has zero-average- see [11]). On the second through fifth lemmas, we estimate energy norm quantities in terms of these quantities and the right-hand side of the perturbation equations. The remaining lemmas bound the right-hand side quantities, leaving us with sums of energy norm terms, initial data, and forcing, all of which can be controlled appropriately.

Lemma 3.1. $\overline{h} = 0$ *and* $\|h\|^2_{L^2(\Omega)} \leqslant C \|Dh\|^2_{L^2(\Omega)}$, *where* C *is a time-independent constant.*

Proof of Lemma (3.1). Recall that h denotes the perturbed height. Let us denote the height in (2.1) by h_*. Integrating the height equation in (2.1) in space, then again in time, yields that

$$\int_\Omega h_*(\mathbf{x},t)\,d\mathbf{x} - \int_\Omega h_*(\mathbf{x},0)\,d\mathbf{x} + \int_0^t \int_\Omega \nabla \cdot (h_*(\mathbf{x},\tau)\mathbf{u}(\mathbf{x},\tau))\,d\mathbf{x}d\tau = 0.$$

Plugging in $h_* = h + \hat{h}$, we have that

$$\int_\Omega h(\mathbf{x},t)\,d\mathbf{x} - \int_\Omega h(\mathbf{x},0)\,d\mathbf{x} + \int_0^t \int_\Omega \nabla \cdot ((h(\mathbf{x},\tau) + \hat{h}(\mathbf{x}))\mathbf{u}(\mathbf{x},\tau))\,d\mathbf{x}d\tau = 0.$$

Applying the divergence theorem and our boundary conditions simplifies the above to

$$\int_\Omega h(\mathbf{x},t)\,d\mathbf{x} - \int_\Omega h(\mathbf{x},0)\,d\mathbf{x} = 0.$$

Since

$$\int_\Omega h(\mathbf{x},0)\,d\mathbf{x} = \int_\Omega (h_0(\mathbf{x}) - \hat{h}(\mathbf{x}))\,d\mathbf{x} = 0,$$

we have that

$$\int_\Omega h(\mathbf{x},t)\,d\mathbf{x} = 0.$$

So, the perturbation h has zero average. Given this information, a direct application from [11] (page 157) allows the result to follow. □

The next lemma provides an estimate on the norms of the perturbed height field and its time integral. We will show this lemma in full detail, since various subsequent lemmas will contain similar arguments as parts of it, which we will omit.

Lemma 3.2.

$$\|Dh\|_4^2 + \int\limits_0^t \|Dh(\tau)\|_4^2\, d\tau \leqslant C\bigg(\|Dh_0\|_4^2 + \|D\mathbf{u}_0\|_3^2 + N^3(0,t) + \int\limits_0^t \left(\lambda \|D\mathbf{u}(\tau)\|_7^2 + \|R_0(\tau)\|_5^2 + \|\mathbf{R}(\tau)\|_4^2\right) d\tau \bigg),$$

where C is a time-independent constant.

Proof of Lemma (3.2)*.* It suffices to show that

$$\left\|D^{k+1}h\right\|^2 + \int\limits_0^t \left\|D^{k+1}h(\tau)\right\|^2 d\tau \leqslant C(\left\|D^{k+1}h_0\right\|^2 + \left\|D^k\mathbf{u}_0\right\|^2 + N^3(0,t) + \int\limits_0^t (\lambda \left\|D^{k+4}\mathbf{u}(\tau)\right\|^2 + \left\|D^{k+1}R_0(\tau)\right\|^2 + \left\|D^k\mathbf{R}(\tau)\right\|^2) d\tau),$$

for $k = 0, 1, 2, 3, 4$.

We begin by taking a gradient of the D^k of the left-hand side of the h perturbation equation, taking its inner product with the $D^k\nabla h$, adding this to the inner product of the D^k of the left-hand side of the $\mathbf{u}$ perturbation equation and $D^k\nabla h$, then integrating in time.

Performing the above, we have that

$$\int\limits_0^t \int\limits_\Omega \left(D^k(\nabla h) \cdot D^k\left(\nabla L_0 + \mathbf{L}\right)\right) d\mathbf{x}d\tau$$

$$= \int\limits_0^t \int\limits_\Omega \left(D^k(\nabla h) \cdot (D^k\nabla(h_\tau + \mathbf{u}\cdot\nabla h + \overline{h}_0\nabla\cdot\mathbf{u})) + D^k(\mathbf{u}_\tau + \lambda\nabla^4\mathbf{u} + \nabla h)\right) d\mathbf{x}d\tau$$

$$= \frac{1}{2}\int\limits_\Omega \int\limits_0^t \frac{\partial}{\partial t}(D^k(\nabla h)\cdot D^k(\nabla h))\, d\tau d\mathbf{x} + \int\limits_0^t \int\limits_\Omega (D^k(\nabla h)\cdot D^k(\nabla h))\, d\mathbf{x}d\tau + \int\int(1)$$

$$= \frac{1}{2}\left\|D^{k+1}h\right\|^2_{L^2(\Omega)} - \frac{1}{2}\left\|D^{k+1}h_0\right\|^2_{L^2(\Omega)} + \int_0^t \left\|D^{k+1}h\right\|^2_{L^2(\Omega)} + \int\int(1),$$

where

$$\int\int(1) = \int_0^t\int_\Omega \left(D^k(\nabla h)\cdot\left(D^k(\nabla(\mathbf{u}\cdot\nabla h) + \overline{h}_0\nabla(\nabla\cdot\mathbf{u}) + \lambda\nabla^4\mathbf{u} + \mathbf{u}_\tau)\right)\right) d\mathbf{x}d\tau.$$

Note that

$$\int_0^t\int_\Omega \left(D^k(\nabla h)\cdot D^k\left(\nabla L_0 + \mathbf{L}\right)\right) d\mathbf{x}d\tau = \int_0^t\int_\Omega \left(D^k(\nabla h)\cdot D^k\left(\nabla R_0 + \mathbf{R}\right)\right) d\mathbf{x}d\tau.$$

Replacing all occurrences of L_0 and $\mathbf{L}$ by R_0 and $\mathbf{R}$, respectively, and rearranging a few terms yields

$$\begin{aligned}
&\frac{1}{2}\left\|D^{k+1}h\right\|^2_{L^2(\Omega)} + \int_0^t \left\|D^{k+1}h\right\|^2_{L^2(\Omega)} \\
&= \frac{1}{2}\left\|D^{k+1}h_0\right\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \left((\nabla(D^kR_0)\cdot D^k(\nabla h) + D^k\mathbf{R}\cdot D^k(\nabla h)\right) d\mathbf{x}d\tau - \int\int(1) \\
&\leqslant \frac{1}{2}\left\|D^{k+1}h_0\right\|^2_{L^2(\Omega)} + \Bigg|\int_0^t\int_\Omega \left((\nabla(D^kR_0)\cdot D^k(\nabla h) + D^k\mathbf{R}\cdot D^k(\nabla h)\right) d\mathbf{x}d\tau - \int\int(1)\Bigg| \\
&\leqslant \frac{1}{2}\left\|D^{k+1}h_0\right\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \left|D^k(\nabla h)\right|\Bigg(\left|D^k(\nabla R_0)\right| + \left|D^k(\nabla\mathbf{R})\right| \\
&+ \left|D^k(\overline{h}_0\nabla(\nabla\cdot\mathbf{u}) + \lambda\nabla^4\mathbf{u})\right|\Bigg) d\mathbf{x}d\tau + \left|\int_0^t\int_\Omega D^k(\nabla h)\cdot D^k(\nabla(\mathbf{u}\cdot\nabla h))\ d\mathbf{x}d\tau\right| \\
&+ \left|\int_0^t\int_\Omega D^k(\nabla h)\cdot D^k\mathbf{u}_\tau\ d\mathbf{x}d\tau\right|.
\end{aligned}$$

We will hereafter refer to the last two integrals above as I_1 and I_2, respectively.

Applying Young's Inequality (with $\epsilon > 0$ arbitrary), we have that, in particular,

$$
\begin{aligned}
&\frac{1}{2}\left\|D^{k+1}h\right\|^2_{L^2(\Omega)} + \int_0^t \left\|D^{k+1}h\right\|^2_{L^2(\Omega)} \\
&\leqslant \frac{1}{2}\left\|D^{k+1}h_0\right\|^2_{L^2(\Omega)} + 2\epsilon \int_0^t \int_\Omega |D^k(\nabla h)|^2 \; d\mathbf{x}d\tau \\
&+ \frac{1}{2\epsilon}\int_0^t \int_\Omega \left(|D^k(\nabla R_0)|^2 + |D^k\mathbf{R}|^2 + |\overline{h}_0 D^k(\nabla(\nabla\cdot\mathbf{u}))|^2 + |\lambda D^k \nabla^4 \mathbf{u}|^2\right) d\mathbf{x}d\tau + I_1 + I_2 \\
&\leqslant \frac{1}{2}\left\|D^{k+1}h_0\right\|^2_{L^2(\Omega)} + 2\epsilon \int_0^t \left\|D^{k+1}h\right\|^2_{L^2(\Omega)} \; d\tau \\
&+ \frac{1}{2\epsilon}\int_0^t \left(\left\|D^{k+1}R_0\right\|^2_{L^2(\Omega)} + \left\|D^k\mathbf{R}\right\|^2_{L^2(\Omega)} + \overline{h}_0\left\|D^{k+2}\mathbf{u}\right\|^2_{L^2(\Omega)} + \lambda\left\|D^{k+4}\mathbf{u}\right\|^2_{L^2(\Omega)}\right) d\tau \\
&+ I_1 + I_2.
\end{aligned}
\tag{3.7}
$$

Before proceeding, we need estimates on I_2 and I_2. Namely, we will demonstrate that

$$
I_1 = \Big|\int_0^t \int_\Omega D^k(\nabla h)\cdot D^k(\nabla(\mathbf{u}\cdot\nabla h)) \; d\mathbf{x}d\tau\Big| \leqslant K_1 N^3(0,t)
$$

and

$$
\begin{aligned}
I_2 = \Big|\int_0^t \int_\Omega D^k(\nabla h)\cdot D^k(\mathbf{u}_\tau) \; d\mathbf{x}d\tau\Big| &\leqslant \epsilon\left\|D^{k+1}h\right\|^2_{L^2(\Omega)} + \frac{K_2}{\epsilon}\big(\left\|D^{k+1}h_0\right\|^2_{L^2(\Omega)} \\
&+ \left\|D^k\mathbf{u}_0\right\|^2_{L^2(\Omega)} + \left\|D^k\mathbf{u}\right\|^2_{L^2(\Omega)}\big) \\
&+ K_3\int_0^t \left(\left\|D^k\mathbf{u}(\tau)\right\|^2_1 + \left\|D^{k+1}R_0(\tau)\right\|^2_{L^2(\Omega)}\right) d\tau \\
&+ K_4 N^3(0,t),
\end{aligned}
$$

where $\epsilon > 0$ comes from applying Young's Inequality and K_1, K_2, K_3, K_4 are time-independent constants.

We apply the Leibniz Rule to I_1:

$$I_1 \leqslant \sum_{l=0}^{k} \binom{k}{l} \Big| \int_0^t \int_\Omega \big(D^k(\nabla h) \cdot D^{k-l} \nabla h D^l(\nabla \mathbf{u}) + D^k(\nabla h) \cdot D^l \mathbf{u} D^{k-l}(\nabla^2 h) \big) \ d\mathbf{x} d\tau \Big|$$

$$\leqslant \sum_{l=0}^{k} \binom{k}{l} \Big| \int_0^t \int_\Omega \Big(\sup_{\mathbf{x}\in\Omega} |D^{l+1}\mathbf{u}| D^k(\nabla h) D^{k-l} \nabla h + \sup_{\mathbf{x}\in\Omega} |D^l \mathbf{u}| D^k(\nabla h) D^{k-l}(\nabla^2 h) \Big) \ d\mathbf{x} d\tau \Big|$$

Next, using Young's Inequality with $\epsilon = 1$,

$$\sum_{l=0}^{k} \binom{k}{l} \Big| \int_0^t \int_\Omega \Big(\sup_{\mathbf{x}\in\Omega} |D^{l+1}\mathbf{u}| D^k(\nabla h) D^{k-l} \nabla h + \sup_{\mathbf{x}\in\Omega} |D^l \mathbf{u}| D^k(\nabla h) D^{k-l}(\nabla^2 h) \Big) \ d\mathbf{x} d\tau \Big|$$

$$\leqslant \sum_{l=0}^{k} \binom{k}{l} \Big| \int_0^t \int_\Omega \big(\sup_{\mathbf{x}\in\Omega} |D^{l+1}\mathbf{u}| ((D^k(\nabla h) \cdot D^k(\nabla h))/2 + (D^{k-l}(\nabla h) \cdot D^{k-l}(\nabla h))/2)$$

$$+ \sup_{\mathbf{x}\in\Omega} |D^l \mathbf{u}| ((D^k(\nabla h) \cdot D^k(\nabla h))/2 + (D^{k-l}(\nabla^2 h) \cdot D^{k-l}(\nabla^2 h))/2) \big) \ d\mathbf{x} d\tau \Big|$$

$$\leqslant \sum_{l=0}^{k} \binom{k}{l} \int_0^t \Big(\sup_{\mathbf{x}\in\Omega} |D^{l+1}\mathbf{u}| \big(\left\| D^{k+1} h \right\|_{L^2(\Omega)}^2 /2 + \left\| D^{k-1+1} h \right\|_{L^2(\Omega)}^2 /2 \big)$$

$$+ \sup_{\mathbf{x}\in\Omega} |D^l \mathbf{u}| \big(\left\| D^{k+1} h \right\|_{L^2(\Omega)}^2 /2 + \left\| D^{k-1+2} h \right\|_{L^2(\Omega)}^2 /2 \big) \Big) \ d\tau.$$

Utilizing the Sobolev embedding of $H^2(\Omega)$ into $C^\alpha(\overline{\Omega})$, we know that a constant A exists for which $\sup_{\mathbf{x}\in\Omega} |D^{l+1}\mathbf{u}| \leqslant A \left\| D^{l+1}\mathbf{u} \right\|_2$. Similarly, a constant B exists for which $\sup_{\mathbf{x}\in\Omega} |D^l \mathbf{u}| \leqslant B \left\| D^l \mathbf{u} \right\|_2$. So, exercising the above and taking the supremum over t,

$$\sum_{l=0}^{k} \binom{k}{l} \int_0^t \Big(\sup_{\mathbf{x}\in\Omega} |D^{l+1}\mathbf{u}| \big(\left\| D^{k+1} h \right\|_{L^2(\Omega)}^2 /2 + \left\| D^{k-1+1} h \right\|_{L^2(\Omega)}^2 /2 \big)$$

$$+ \sup_{\mathbf{x}\in\Omega} |D^l \mathbf{u}| \big(\left\| D^{k+1} h \right\|_{L^2(\Omega)}^2 /2 + \left\| D^{k-1+2} h \right\|_{L^2(\Omega)}^2 /2 \big) \Big) \ d\tau$$

$$\leqslant \sum_{l=0}^{k} \binom{k}{l} \Big(A \sup_{\tau\in[0,t]} \left\| D^{l+1}\mathbf{u} \right\|_2 \int_0^t \big(\left\| D^{k+1} h \right\|_{L^2(\Omega)}^2 + \left\| D^{k-l+1} h \right\|_{L^2(\Omega)}^2 \big) \ d\tau$$

$$+ B \sup_{\tau\in[0,t]} \left\|D^l \mathbf{u}\right\|_2 \int_0^t \left(\left\|D^{k+1}h\right\|^2_{L^2(\Omega)} + \left\|D^{k-l+2}h\right\|^2_{L^2(\Omega)} \; d\tau \right)\Bigg)$$

$$\leqslant K_1 N(0,t) N^2(0,t) = K_1 N^3(0,t),$$

where K_1 is a time-independent constant.

The above is valid of all combinations of l and k, where k goes up to 4, except for the cases where $(l, k) = (0, 4), (4, 4)$. When $(l, k) = (0, 4)$, the second term (the problem term) in the Leibniz rule becomes

$$\Big|\int_0^t \int_\Omega D^4(\nabla h)\cdot \mathbf{u} D^4(\nabla^2 h)\; d\mathbf{x}d\tau\Big| = \frac{1}{2}\Big|\int_0^t \int_\Omega \nabla\cdot (D^4(\nabla h) D^4(\nabla h))\cdot \mathbf{u}\; d\mathbf{x}d\tau\Big|.$$

Integrating by parts, we have that

$$\frac{1}{2}\Big|\int_0^t \int_\Omega \nabla(D^4(\nabla h)\cdot D^4(\nabla h))\cdot \mathbf{u}\; d\mathbf{x}d\tau\Big|$$

$$= \frac{1}{2}\Big|\int_0^t \int_{\partial\Omega} (D^k(\nabla h))^2 \mathbf{u}\cdot \mathbf{n}\; d\mathbf{x}d\tau - \int_0^t \int_\Omega (D^k(\nabla h)\cdot D^k(\nabla h))\nabla\cdot \mathbf{u}\; d\mathbf{x}d\tau\Big|,$$

where $\mathbf{n}$ denotes the outward unit normal over Ω. The integration over $\partial\Omega$ vanishes from the periodic boundaries. Performing the same supremum process as done earlier ensures that we obtain a $\sup_{\mathbf{x}\in\Omega}|D\mathbf{u}|$, without violating how many derivatives we are allowed on h.

When $(l, k) = (4, 4)$, we encounter a problem on the term $\sup_{\tau\in[0,t]} \left\|D^{l+1}\mathbf{u}\right\|_2$. In order to assuage this issue, which comes from the first term in the Leibniz Rule, we pull out the supremum $\sup_{\mathbf{x}} |Dh|$, and we apply Young's Inequality to $D^4(\nabla h)$ and $D^4(\nabla \mathbf{u})$, then continue in the same manner. Thus, the estimate on I_1 holds.

Next, we prove the estimate on I_2. Integrating by parts in t and utilizing the triangle inequality yields

$$\Big|\int_0^t \int_\Omega D^k(\nabla h)\cdot D^k\mathbf{u}_\tau \, d\mathbf{x}d\tau\Big|$$

$$\leqslant \Big|\int_\Omega D^k(\nabla h(t))\cdot D^k\mathbf{u}(t)\, d\mathbf{x}\Big| + \Big|\int_\Omega D^k(\nabla h(0))\cdot D^k\mathbf{u}(0)\, d\mathbf{x}\Big|$$

$$+\Big|\int_0^t \int_\Omega \left(\frac{\partial}{\partial \tau} D^k(\nabla h(\tau))\right)\cdot D^k\mathbf{u}(\tau)\, d\mathbf{x}d\tau\Big|.$$

Using Young's Inequality on the first two terms and substituting the re-arranged h perturbation equation in for h_t yields that, in particular,

$$I_2 \leqslant \frac{\epsilon}{2}\left\|D^{k+1}h(t)\right\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\left\|D^k\mathbf{u}(t)\right\|^2_{L^2(\Omega)} + \frac{\epsilon}{2}\left\|D^{k+1}h_0\right\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\left\|D^k\mathbf{u}_0\right\|^2_{L^2(\Omega)}$$

$$+\Big|\int_0^t \int_\Omega D^k\mathbf{u}\cdot D^k\nabla(R_0 - \mathbf{u}\cdot\nabla h - \overline{h_0}\nabla\cdot\mathbf{u})\, d\mathbf{x}d\tau\Big|$$

$$\leqslant \frac{\epsilon}{2}\left\|D^{k+1}h(t)\right\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\left\|D^k\mathbf{u}(t)\right\|^2_{L^2(\Omega)} + \left\|D^{k+1}h_0\right\|^2_{L^2(\Omega)} + \left\|D^k\mathbf{u}_0\right\|^2_{L^2(\Omega)}$$

$$+\Big|\int_0^t \int_\Omega D^k\mathbf{u}\cdot D^k\nabla(R_0 - \mathbf{u}\cdot\nabla h - \overline{h_0}\nabla\cdot\mathbf{u})\, d\mathbf{x}d\tau\Big|$$

Now, note that, using Young's Inequality ($\epsilon = 1$) and integrating by parts,

$$\Big|\int_0^t \int_\Omega D^k\mathbf{u}\cdot D^k\nabla(R_0 - \mathbf{u}\cdot\nabla h - \overline{h_0}\nabla\cdot\mathbf{u})\, d\mathbf{x}d\tau\Big|$$

$$\leqslant \int_0^t \int_\Omega \left(\frac{1}{2}(D^k\mathbf{u})\cdot(D^k\mathbf{u}) + \frac{1}{2}((D^k(\nabla R_0)\cdot(D^k(\nabla R_0))\right)\, d\mathbf{x}d\tau$$

$$+\Big|\int_0^t \int_\Omega D^k\mathbf{u}\cdot D^k(\nabla(\mathbf{u}\cdot\nabla h))\, d\mathbf{x}d\tau\Big| + \Big|\overline{h_0}\int_0^t \int_{\partial\Omega} D^k(\nabla\mathbf{u})D^k\mathbf{u}\cdot\mathbf{n}\, d\mathbf{x}d\tau$$

$$-\overline{h_0}\int_0^t \int_\Omega D^k(\nabla\mathbf{u})\cdot D^k(\nabla\mathbf{u})\, d\mathbf{x}d\tau\Big|$$

$$
\leqslant \int_0^t \Big(\left\|D^k\mathbf{u}(\tau)\right\|_{L^2(\Omega)}^2 + \left\|D^{k+1}R_0(\tau)\right\|_{L^2(\Omega)}^2
$$

$$
+ \Big| \int_\Omega D^k\mathbf{u} \cdot D^k(\nabla(\mathbf{u}\cdot\nabla h))\, d\mathbf{x} \Big| + \overline{h_0} \left\|D^{k+1}\mathbf{u}(\tau)\right\|_{L^2(\Omega)}^2 \Big)\, d\tau,
$$

where the integral term over $\partial\Omega$ vanishes from the boundary conditions. Additionally, by utilizing the Leibniz Rule as earlier, the integral term is bounded by $AN^3(0,t)$, for some time-independent constant A. Thus,

$$
I_2 \leqslant \frac{\epsilon}{2}\left\|D^{k+1}h(t)\right\|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon}\left\|D^k\mathbf{u}(t)\right\|_{L^2(\Omega)}^2 + \frac{\epsilon}{2}\left\|D^{k+1}h_0\right\|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon}\left\|D^k\mathbf{u}_0\right\|_{L^2(\Omega)}^2
$$

$$
+ \Big| \int_0^t \int_\Omega D^k\mathbf{u}\cdot D^k\nabla(R_0 - \mathbf{u}\cdot\nabla h - \overline{h_0}\nabla\cdot\mathbf{u})\, d\mathbf{x} d\tau \Big|
$$

$$
\leqslant \frac{\epsilon}{2}\left\|D^{k+1}h(t)\right\|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon}\left\|D^k\mathbf{u}(t)\right\|_{L^2(\Omega)}^2 + \frac{\epsilon}{2}\left\|D^{k+1}h_0\right\|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon}\left\|D^k\mathbf{u}_0\right\|_{L^2(\Omega)}^2
$$

$$
+ \int_0^t \left(\left\|D^k\mathbf{u}(\tau)\right\|_{L^2(\Omega)}^2 + \left\|D^{k+1}R_0(\tau)\right\|_{L^2(\Omega)}^2 + \overline{h_0}\left\|D^{k+1}\mathbf{u}(\tau)\right\|_{L^2(\Omega)}^2 \right) d\tau + AN^3(0,t).
$$

Choosing constants appropriately yields that

$$
I_2 \leqslant \frac{\epsilon}{2}\left\|D^{k+1}h(t)\right\|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon}\left\|D^k\mathbf{u}(t)\right\|_{L^2(\Omega)}^2 + \left\|D^{k+1}h_0\right\|_{L^2(\Omega)}^2 + \left\|D^k\mathbf{u}_0\right\|_{L^2(\Omega)}^2
$$

$$
+ \int_0^t \left(\left\|D^k\mathbf{u}(\tau)\right\|_{L^2(\Omega)}^2 + \left\|D^{k+1}R_0(\tau)\right\|_{L^2(\Omega)}^2 + \overline{h_0}\left\|D^{k+1}\mathbf{u}(\tau)\right\|_{L^2(\Omega)}^2 \right) d\tau + AN^3(0,t)
$$

$$
\leqslant \epsilon\left\|D^{k+1}h(\tau)\right\|_{L^2(\Omega)}^2 + \frac{K_2}{\epsilon}\left(\left\|D^{k+1}h_0\right\|_{L^2(\Omega)}^2 + \left\|D^k\mathbf{u}_0\right\|_{L^2(\Omega)}^2 + \left\|D^k\mathbf{u}(t)\right\|_{L^2(\Omega)}^2 \right)
$$

$$
+ K_3\int_0^t \left(\left\|D^k\mathbf{u}(\tau)\right\|_1^2 + \left\|D^{k+1}R_0(\tau)\right\|_{L^2(\Omega)}^2 \right) d\tau + K_4N^3(0,t),
$$

where K_2, K_3, K_4 are time-independent constants. We have established the required estimates. Plugging them into (3.7),

$$
\frac{1}{2}\left\|D^{k+1}h\right\|_{L^2(\Omega)}^2 + \int_0^t \left\|D^{k+1}h\right\|_{L^2(\Omega)}^2 d\tau \leqslant \frac{1}{2}\left\|D^{k+1}h_0\right\|_{L^2(\Omega)}^2 + 2\epsilon\int_0^t \left\|D^{k+1}h\right\|_{L^2(\Omega)}^2\, d\tau
$$

$$+\frac{1}{2\epsilon}\int_0^t \left(\left\|D^{k+1}R_0\right\|^2_{L^2(\Omega)} + \left\|D^k\mathbf{R}\right\|^2_{L^2(\Omega)} + \overline{h}_0\left\|D^{k+2}\mathbf{u}\right\|^2_{L^2(\Omega)} + \lambda\left\|D^{k+4}\mathbf{u}\right\|^2_{L^2(\Omega)}\right)\, d\tau$$

$$+ I_1 + I_2$$

$$\leqslant \frac{1}{2}\left\|D^{k+1}h_0\right\|^2_{L^2(\Omega)} + 2\epsilon\int_0^t \left\|D^{k+1}h\right\|^2_{L^2(\Omega)}\, d\tau + \frac{1}{2\epsilon}\int_0^t \Bigg(\left\|D^{k+1}R_0\right\|^2_{L^2(\Omega)} + \left\|D^k\mathbf{R}\right\|^2_{L^2(\Omega)}$$

$$\overline{h}_0\left\|D^{k+2}\mathbf{u}\right\|^2_{L^2(\Omega)} + \lambda\left\|D^{k+4}\mathbf{u}\right\|^2_{L^2(\Omega)}\Bigg)\, d\tau + K_1N^3(0,t) + \epsilon\left\|D^{k+1}h(\tau)\right\|^2_{L^2(\Omega)}$$

$$+ \frac{K_2}{\epsilon}\left(\left\|D^{k+1}h_0\right\|^2_{L^2(\Omega)} + \left\|D^k\mathbf{u}_0\right\|^2_{L^2(\Omega)} + \left\|D^k\mathbf{u}(t)\right\|^2_{L^2(\Omega)}\right)$$

$$+ K_3\int_0^t \left(\left\|D^k\mathbf{u}(\tau)\right\|^2_1 + \left\|D^{k+1}R_0(\tau)\right\|^2_{L^2(\Omega)}\right)\, d\tau + K_4N^3(0,t).$$

Now, we merely re-arrange terms, choose epsilons appropriately, choose a constant C large enough, then sum over the k's. □

Next, we establish an estimate on the norm of $\mathbf{u}$ and its time integral.

Lemma 3.3.

$$\|h\|_5^2 + \|\mathbf{u}\|_6^2 + \lambda\int_0^t \|D\mathbf{u}\|_7^2\, d\tau \leqslant C\Bigg(\|h_0\|_5^2 + \|\mathbf{u}_0\|_6^2 + N^3(0,t)$$

$$+ \int_0^t \left(\|R_0\|_5^2 + \|\mathbf{R}\|_6^2 + \epsilon\,\|Dh\|_4^2\, d\tau\right)\Bigg),$$

where C is a time-independent constant, and $\epsilon > 0$ comes from Young's Inequality.

Proof of Lemma (3.3). For $k = 0,1,2,3,4$ and 5, we will take the inner product of D^kh and D^kL_0, add it to the inner product of $D^k\mathbf{u}$ and $\overline{h}_0D^k\mathbf{L}$, then integrate over time. So, let $k \in \{0,1,2,3,4,5\}$ be arbitrary. Then, in a process similar to Lemma 2, we compute

$$
\begin{aligned}
&\int_0^t \int_\Omega \left(D^k h D^k L_0 + D^k \mathbf{u} \cdot D^k \mathbf{L}\right) \, d\mathbf{x} d\tau \\
&= \int_0^t \int_\Omega \left(D^k h D^k \left(h_\tau + \mathbf{u} \cdot \nabla h + \overline{h}_0 \nabla \cdot \mathbf{u}\right) + D^k \mathbf{u} \cdot \overline{h}_0 D^k \left(\mathbf{u}_\tau + \lambda \nabla^4 \mathbf{u} + \nabla h\right)\right) \, d\mathbf{x} d\tau \\
&= \frac{1}{2} \left(\left\| D^k h \right\|^2_{L^2(\Omega)} - \left\| D^k h_0 \right\|^2_{L^2(\Omega)} + \left\| D^k \mathbf{u} \right\|^2_{L^2(\Omega)} - \left\| D^k \mathbf{u}_0 \right\|^2_{L^2(\Omega)} \right) \\
&+ \int_0^t \int_\Omega \left(D^k h D^k \left(\mathbf{u} \cdot \nabla h + \overline{h}_0 \nabla \cdot \mathbf{u}\right) + D^k \mathbf{u} \cdot \overline{h}_0 D^k \left(\lambda \nabla^4 \mathbf{u} + \nabla h\right)\right) \, d\mathbf{x} d\tau \\
&= \int_0^t \int_\Omega \left(D^k h D^k R_0 + D^k \mathbf{u} \cdot \overline{h}_0 D^k \mathbf{R}\right) \, d\mathbf{x} d\tau.
\end{aligned}
$$

Note that the terms $\int_0^t \int_\Omega D^k h D^k (\overline{h}_0 \nabla \cdot \mathbf{u}) \, d\mathbf{x} d\tau$ and $\int_0^t \int_\Omega D^k \mathbf{u} \cdot \overline{h}_0 D^k (\nabla h) \, d\mathbf{x} d\tau$ will cancel each other after integrating one of them by parts over Ω. Also, integrating the biharmonic term by parts twice and applying the periodic boundary conditions yields that

$$
\lambda \int_0^t \int_\Omega D^k \mathbf{u} D^k \nabla^4 \mathbf{u} d\mathbf{x} d\tau = \lambda \int_0^t \int_\Omega D^k \nabla^2 \mathbf{u} \cdot D^k \nabla^2 \mathbf{u} d\mathbf{x} d\tau = \lambda \int_0^t \left\| D^k \nabla^2 \mathbf{u} \right\|^2_{L^2(\Omega)} d\tau.
$$

Re-arranging terms and scaling yields, for some time-independent constant C, that

$$
\begin{aligned}
&\left\| D^k h \right\|^2_{L^2(\Omega)} + \left\| D^k \mathbf{u} \right\|^2_{L^2(\Omega)} + \lambda \int_0^t \left\| \nabla^2 \mathbf{u} \right\|^2_{L^2(\Omega)} d\tau \\
&\leqslant C \Bigg(\left\| D^k h_0 \right\|^2_{L^2(\Omega)} + \left\| D^k \mathbf{u}_0 \right\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \left(|D^k h| \left(|D^k (\mathbf{u} \cdot \nabla h)| + |D^k R_0| \right) \right. \\
&\left. + |D^k \mathbf{u} \cdot D^k \mathbf{R}| \right) \, dx x d\tau \Bigg).
\end{aligned}
$$

For the first integral term, we follow a similar process to that in Lemma 2, integrating by parts for higher k to reduce the order of integration on h. As such, the term will be bounded by $KN^3(0,t)$, some time-independent constant K. For the other two terms, we will use Young's Inequality for $\epsilon_1, \epsilon_2 > 0$. Applying the above, we obtain that

$$\begin{aligned}
&\left\|D^k h\right\|^2_{L^2(\Omega)} + \left\|D^k \mathbf{u}\right\|^2_{L^2(\Omega)} + \lambda \int_0^t \left\|D^k \nabla^2 \mathbf{u}\right\|^2_{L^2(\Omega)} d\tau \\
&\leqslant C\Bigg(\left\|D^k h_0\right\|^2_{L^2(\Omega)} + \left\|D^k \mathbf{u}_0\right\|^2_{L^2(\Omega)} + N^3(0,t) \\
&+ \int_0^t \left(\frac{\epsilon_1}{2} \left\|D^k h\right\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon_1} \left\|D^k R_0\right\|_{L^2(\Omega)}\right) d\tau \\
&+ \int_0^t \left(\frac{\epsilon_2}{2} \left\|D^k \mathbf{u}\right\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon_2} \left\|D^k \mathbf{R}\right\|^2_{L^2(\Omega)}\right) d\tau\Bigg).
\end{aligned}$$

Summing from $k = 0$ to $k = 5$ and applying the Poincaré Inequality on $\mathbf{u}$,

$$\begin{aligned}
&\|h\|^2_5 + \|\mathbf{u}\|^2_5 + \lambda \int_0^t \left\|\nabla^2 \mathbf{u}\right\|^2_5 d\tau \\
&\leqslant C\Bigg(\|h_0\|^2_5 + \|\mathbf{u}_0\|^2_5 + N^3(0,t) + \int_0^t \left(\frac{\epsilon_1}{2} \|h\|^2_5 + \frac{1}{2\epsilon_1} \|R_0\|^2_5\right) d\tau \\
&+ \int_0^t \left(\frac{\epsilon_2}{2} \|D\mathbf{u}\|^2_5 + \frac{1}{2\epsilon_2} \|\mathbf{R}\|^2_5\right) d\tau\Bigg).
\end{aligned}$$

Applying the elliptic estimate present in Corollary (A.1), taking an appropriate choice of ϵ_2, and making terms larger allow us to conclude that

$$\begin{aligned}
&\|h\|^2_5 + \|\mathbf{u}\|^2_5 + \lambda \int_0^t \|D\mathbf{u}\|^2_6 d\tau \\
&\leqslant C\Bigg(\|h_0\|^2_5 + \|\mathbf{u}_0\|^2_5 + N^3(0,t) + \int_0^t \left(\frac{\epsilon_1}{2} \|h\|^2_5 + \frac{1}{2\epsilon_1} \|R_0\|^2_5\right) d\tau + \int_0^t \|\mathbf{R}\|^2_5 d\tau\Bigg).
\end{aligned}$$

When $k = 6$, we perform a similar process to the above, but only the estimate on the momentum equations, providing us with a similar estimate (in that we pick up no new terms). This calculation is nearly identical to how we estimated the quantity previously, modulo a substitution of a minor re-arrangement of the height equation in (3.1), so we omit it. Combining the results and applying the Poincaré Inequality in h yields that

$$\|h\|_5^2 + \|\mathbf{u}\|_6^2 + \lambda \int_0^t \|D\mathbf{u}\|_7^2 \, d\tau \leqslant C\Bigg(\|h_0\|_5^2 + \|\mathbf{u}_0\|_6^2 + N^3(0,t) + \int_0^t \left(\|R_0\|_5^2 + \|\mathbf{R}\|_6^2 + \epsilon \|Dh\|_4^2 \, d\tau\right) \Bigg).$$

□

Remark 3.5. When we connect the lemmas together, the left-hand side of the inequality will consist of energy norm terms. We will take ϵ appropriately such that $\epsilon \int_0^t \|Dh\|_4 \, d\tau$ can be moved over from the right to the left, then we will rescale, effectively removing this term. This will also be the last estimate used on terms from the left-hand side of (3.1). Hence, without loss of generality, we will remove this term when applying the above lemma. △

The next lemma obtains an estimate on the time derivative of the height and its time integral.

Lemma 3.4.

$$\|h_t\|_5^2 \leqslant C_1 \left(\|R_0\|_5^2 + N^3(0,t) + \|D\mathbf{u}\|_5^2\right)$$

and

$$\int_0^t \|h_\tau\|_5^2 \, d\tau \leqslant C_2 \left(\int_0^t (\|D\mathbf{u}\|_7^2 + \|R_0\|_5^2) d\tau + N^3(0,t) \right),$$

where C_1 and C_2 are time-independent constants.

Proof of Lemma (3.4). For the first estimate, let k run from 0 to 5. Then, applying Young's Inequality for $\epsilon > 0$,

$$
\begin{aligned}
\left\|D^k h_t\right\|_{L^2(\Omega)}^2 &= \int_\Omega D^k h_t D^k h_t \; d\mathbf{x} = \int_\Omega D^k h_t D^k (R_0 - \overline{h}_0(\nabla \cdot \mathbf{u}) - \mathbf{u} \cdot \nabla h) \; d\mathbf{x} \\
&\leqslant \frac{3\epsilon}{2} \left\|D^k h_t\right\|_{L^2(\Omega)} + \frac{1}{2\epsilon} \left\|D^k R_0\right\|_{L^2(\Omega)}^2 + \frac{\overline{h}_0}{2\epsilon} \left\|D^{k+1}\mathbf{u}\right\|_{L^2(\Omega)}^2 \\
&+ \Big| \int_\Omega D^k h_t D^k (\mathbf{u} \cdot \nabla h) d\mathbf{x} \Big|.
\end{aligned}
$$

In a manner similar to Lemma (3.2), one can show that

$$
\Big| \int_\Omega D^k h_t D^k (\mathbf{u} \cdot \nabla h) \; d\mathbf{x} \Big| \leqslant K N^3(0,t),
$$

some time-independent constant K, for all $k \in \{0,1,2,3,4,5\}$ (note that it will require integrating by parts for $k = 5$). Hence, by picking epsilon small enough, scaling, and making the $\mathbf{u}$ term larger, we have that

$$
\left\|D^k h_t\right\|_{L^2(\Omega)}^2 \leqslant C_1 \left(\left\|D^k R_0\right\|_{L^2(\Omega)}^2 + N^3(0,t) + \left\|D^{k+1}\mathbf{u}\right\|_{L^2(\Omega)} \right).
$$

Summing over k, we have that

$$
\|h_t\|_5^2 \leqslant C_1 \left(\|R_0\|_5 + N^3(0,t) + \|D\mathbf{u}\|_5 \right).
$$

The time-integrated estimate is performed similarly. □

Now, we do the same with the perturbed velocity.

Lemma 3.5.

$$
\|\mathbf{u}_t\|_2^2 \leqslant C_1 \left(\|\mathbf{R}\|_5^2 + \lambda \|D\mathbf{u}\|_5^2 + N^3(0,t) + \|Dh\|_4^2 \right),
$$

and

$$
\int_0^t \|\mathbf{u}_\tau\|_4^2 \, d\tau \leqslant C_2 \left(\int_0^t (\|\mathbf{R}\|_6^2 + \lambda \|D\mathbf{u}\|_7 + \|Dh\|_4^2) d\tau + N^3(0,t) \right),
$$

where C_1 and C_2 are time-independent constants.

Proof of Lemma (3.5). We proceed as in Lemma (3.4), but here $k \in \{0,1,2\}$ for the first estimate, and through 4 in the second. We apply Young's Inequality with $\epsilon > 0$ to the quantity, obtaining

$$\begin{aligned}\left\|D^k \mathbf{u}_t\right\|_{L^2(\Omega)} &= \int_\Omega (D^k \mathbf{u}_t \cdot D^k \mathbf{u}_t)\ d\mathbf{x} = \int_\Omega D^k \mathbf{u}_t \cdot D^k(\mathbf{R} - \lambda \nabla^4 \mathbf{u} - \nabla h)\ d\mathbf{x} \\ &\leqslant \frac{3\epsilon}{2}\left\|D^k \mathbf{u}_t\right\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\left\|D^k \mathbf{R}\right\|^2_{L^2(\Omega)} + \frac{\lambda}{2\epsilon}\left\|D^k \nabla^4 \mathbf{u}\right\|_{L^2(\Omega)} \\ &+ \frac{1}{2\epsilon}\left\|D^{k+1} h\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Applying this bound, choosing an appropriate ϵ, scaling, applying the Poincaré Inequality to the velocity, and making terms larger leaves us with

$$\left\|D^k \mathbf{u}_t\right\|^2_{L^2(\Omega)} \leqslant C_1 \left(\left\|D^k \mathbf{R}\right\|^2_{L^2(\Omega)} + \lambda \left\|D^{k+1} \mathbf{u}\right\|^2_3 + \left\|D^{k+1} h\right\|^2_2\right),$$

some time-independent constant C_1. Summing over the k's and making terms larger yields the first estimate. The second estimate follows similarly. □

The next two lemmas involve bounding the right-hand side of the h perturbation equation, up to a time t.

Lemma 3.6.

$$\sup_{t \in [0,t]} \|R_0\|^2_5 \leqslant C(N^2(0,t) + \|DV\|^2_5) N^2(0,t),$$

where C is a time-independent constant.

Proof of Lemma (3.6). Utilizing the Banach algebra norm property of $H^5(\Omega)$ and the Poincaré inequality on the quantity $\overline{h}_0 - \hat{h} = \overline{V} - V$,

$$\begin{aligned}\sup_{\tau \in [0,t]} \|R_0\|^2_5 &= \sup_{\tau \in [0,t]} \left\{ \left\|(\overline{h}_0 - h - \hat{h})\nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \hat{h}\right\|^2_5 \right\} \\ &\leqslant \sup_{\tau \in [0,t]} \left\{ \|h \nabla \cdot \mathbf{u}\|^2_5 + \left\|(\overline{h}_0 - \hat{h})\nabla \cdot \mathbf{u}\right\|^2_5 + \left\|\mathbf{u} \cdot \nabla \hat{h}\right\|^2_5 \right\} \\ &\leqslant C \sup_{\tau \in [0,t]} \left\{ \left(\|h\|^2_5 + \left\|\overline{h}_0 - \hat{h}\right\|^2_5 + \left\|\nabla \hat{h}\right\|^2_5\right) \|\mathbf{u}\|^2_6 \right\} \\ &\leqslant C\left(N^2(0,t) + \|DV\|^2_5\right) N^2(0,t),\end{aligned}$$

where C is a time-independent constant dependent the parameters from the Banach Algebra norm property and the Poincaré Inequality. $\square$

Lemma 3.7.

$$\int_0^t \|R_0\|_5^2 \, d\tau \leqslant C(N^2(0,t) + \|DV\|_5^2)N^2(0,t),$$

where C is a time-independent constant.

Proof of Lemma (3.7). We follow a similar process to the above. Utilizing the Banach algebra property of $H^5(\Omega)$ and the Poincaré Inequality on $\mathbf{u}$,

$$\begin{aligned}
\int_0^t \|R_0\|_5^2 \, d\tau &= \int_0^t \left(\left\| (\overline{h}_0 - h - \hat{h})\nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \hat{h} \right\|_5^2 \right) d\tau \\
&\leqslant \int_0^t \left(\|h \nabla \cdot \mathbf{u}\|_5^2 + \left\| (\overline{h}_0 - \hat{h})\nabla \cdot \mathbf{u} \right\|_5^2 + \left\| \mathbf{u} \cdot \nabla \hat{h} \right\|_5^2 \right) d\tau \\
&\leqslant C \int_0^t \left(\left(\|h\|_5^2 + \left\| \overline{h}_0 - \hat{h} \right\|_5^2 + \left\| \nabla \hat{h} \right\|_5^2 \right) \|\mathbf{u}\|_6^2 \right) d\tau \\
&\leqslant C \int_0^t \left(\left(\|h\|_5^2 + \|DV\|_5^2 \right) \|\mathbf{u}\|_5^2 \right) d\tau \\
&\leqslant C \sup_{\tau \in [0,t]} \left\{ \|h\|_5^2 + \|DV\|_5^2 \right\} \int_0^t \|D\mathbf{u}\|_7^2 \, d\tau \\
&\leqslant C \left(N^2(0,t) + \|DV\|_5^2 \right) N^2(0,t),
\end{aligned}$$

where C is a time independent constant depending on parameters stemming from the Poincaré inequality on $\mathbf{u}$ and the Banach algebra norm property on $H^5(\Omega)$. $\square$

Lemma 3.8.

$$\sup_{\tau \in [0,t]} \|\mathbf{R}\|_5^2 \leqslant CN^4(0,t),$$

where C is a time-independent constant.

Proof of Lemma (3.8). Using the Banach algebra norm property of $H^5(\Omega)$, we compute that

$$\begin{aligned}\sup_{\tau\in[0,t]} \|\mathbf{R}\|_5^2 &= \sup_{\tau\in[0,t]} \left\{\|-(\mathbf{u}\cdot\nabla)\mathbf{u}\|_5^2\right\}\\ &\leqslant C \sup_{\tau\in[0,t]} \left\{ \|\mathbf{u}\|_5^2 \, \|D\mathbf{u}\|_5^2 \right\}\\ &\leqslant CN^4(0,t).\end{aligned}$$

where C is a time-independent constant depending on the parameters present in the conditions of the Banach algebra norm property of $H^5(\Omega)$, which proves the result. □

Lemma 3.9.

$$\int_0^t \|\mathbf{R}\|_6^2 \, d\tau \leqslant CN^4(0,t),$$

where C is a time-independent constant.

Proof of Lemma (3.9)*.* Utilizing the Banach algebra property of $H^6(\Omega)$,

$$\begin{aligned}\int_0^t \|\mathbf{R}\|_6^2 \, d\tau &= \int_0^t \|-(\mathbf{u}\cdot\nabla)\mathbf{u}\|_6^2 \, d\tau\\ &\leqslant C \int_0^t \|\mathbf{u}\|_6^2 \, \|D\mathbf{u}\|_7^2 \, d\tau\\ &\leqslant C \sup_{\tau\in[0,t]} \|\mathbf{u}\|_6^2 \int_0^t \|D\mathbf{u}\|_7^2 \, d\tau\\ &\leqslant CN^4(0,t),\end{aligned}$$

where C is a time-independent constant that depends on the parameters present in the conditions of the Banach algebra norm property of $H^6(\Omega)$. □

We are ready to prove Proposition (3.3).

Proof of Proposition (3.3). Consider the quantity

$$E(t) = \|h\|_5^2+\|h_t\|_5^2+\|\mathbf{u}\|_6^2+\|\mathbf{u}_t\|_2^2+\int_0^t \left(\|Dh(\tau)\|_4^2 + \|h_\tau(\tau)\|_5^2 + \|D\mathbf{u}(\tau)\|_7^2 + \|\mathbf{u}_\tau(\tau)\|_4^2\right) d\tau.$$

Applying, in order, Lemmas (3.4), (3.5), (3.2), and (3.3) (with Lemma (3.1) used throughout) and making terms larger when necessary,

$$\begin{aligned}
E(t) &\leqslant C\Bigg(\|Dh\|_4^2 + \|h_t\|_5^2 + \|\mathbf{u}\|_6^2 + \|\mathbf{u}_t\|_2^2 \\
&+ \int_0^t \left(\|Dh(\tau)\|_4^2 + \|h_\tau(\tau)\|_5^2 + \|D\mathbf{u}(\tau)\|_7^2 + \|\mathbf{u}_\tau(\tau)\|_4^2\right) d\tau \Bigg) \\
&\leqslant C\Bigg(\|Dh\|_4^2 + \|R_0\|_5^2 + \|\mathbf{u}\|_6^2 + \|\mathbf{u}_t\|_2^2 + N^3(0,t) \\
&+ \int_0^t \left(\|Dh(\tau)\|_4^2 + \|R_0(\tau)\|_5^2 + \|D\mathbf{u}(\tau)\|_7^2 + \|\mathbf{u}_\tau(\tau)\|_4^2\right) d\tau \Bigg) \\
&\leqslant C\Bigg(\|Dh\|_4^2 + \|R_0\|_5^2 + \|D\mathbf{u}\|_5^2 + \|\mathbf{R}\|_5^2 + N^3(0,t) \\
&+ \int_0^t \left(\|Dh(\tau)\|_4^2 + \|R_0(\tau)\|_5^2 + \|D\mathbf{u}(\tau)\|_7^2 + \|\mathbf{R}(\tau)\|_6^2\right) d\tau \Bigg) \\
&\leqslant C\Bigg(\|h\|_5^2 + \|Dh_0\|_4^2 + \|D\mathbf{u}_0\|_5^2 + \|R_0\|_5^2 + \|D\mathbf{u}\|_5^2 + \|\mathbf{R}\|_5^2 + N^3(0,t) \\
&+ \int_0^t \left(\|R_0(\tau)\|_5^2 + \|D\mathbf{u}(\tau)\|_7^2 + \|\mathbf{R}(\tau)\|_6^2\right) d\tau \Bigg) \\
&\leqslant C\Bigg(\|Dh_0\|_4^2 + \|D\mathbf{u}_0\|_5^2 + \|R_0\|_5^2 + \|\mathbf{R}\|_5^2 + N^3(0,t) \\
&+ \int_0^t \left(\|R_0(\tau)\|_5^2 + \|\mathbf{R}(\tau)\|_6^2\right) d\tau \Bigg).
\end{aligned}$$

Taking the supremum over the interval $[0,T]$ yields that

$$N^2(0,T) \leqslant C\Bigg(\|Dh_0\|_4^2 + \|D\mathbf{u}_0\|_5^2 + \sup_{t\in[0,T]} \left\{\|R_0(t)\|_5^2 + \|\mathbf{R}(t)\|_5^2\right\} + N^3(0,T) \\ + \int_0^T \left(\|R_0(\tau)\|_5^2 + \|\mathbf{R}(\tau)\|_6^2\right) d\tau\Bigg)$$

From Lemmas (3.6)-(3.9),

$$\begin{aligned} N^2(0,T) &\leqslant C\Bigg(\|Dh_0\|_4^2 + \|D\mathbf{u}_0\|_5^2 + N^3(0,T) + N^2(0,T)\left(N^2(0,T) + \|DV\|_5^2\right)\Bigg) \\ &= C\left(\|Dh_0\|_4^2 + \|D\mathbf{u}_0\|_5^2\right) + CN^2(0,T)\left(N(0,T) + N^2(0,T) + \|DV\|_5^2\right) \end{aligned}$$

Take $\epsilon_1 = \inf\{E_1(\overline{h}_0), E_2(\overline{h}_0), \{z : C(z+2z^2) < 1/2\}\}$ (i.e. making the initial data small enough) and $C_1 = \sqrt{\dfrac{C}{1-C(\epsilon_1+2\epsilon_1^2)}}$. Hence, by re-arranging terms and letting $N(0,T) \leqslant \epsilon_1$ and $\|DV\|_5 \leqslant \epsilon_1$ yields that

$$N^2(0,T) \leqslant C_1^2\left(\|Dh_0\|_4^2 + \|D\mathbf{u}_0\|_5^2\right) \leqslant C_1^2 N^2(0,0).$$

Squaring both sides yields the desired result. □

Now, we are ready to prove the main theorem.

Proof of Theorem (2.1). First, we show that the energy norm is bounded for all time. Take the initial data to be small enough so that

$$N(0,0) \leqslant \min\left\{\epsilon_1, \frac{\epsilon_1}{C_0}, \frac{\epsilon_1}{C_1\sqrt{1+C_0^2}}\right\} = \delta$$

and

$$\|DV\|_5 \leqslant \delta \leqslant \epsilon_1.$$

We will demonstrate that $N(0,n\tau) \leqslant \epsilon_1$ for all $n \in \mathbb{N}$. We proceed inductively. We start with the base case. Let $T = 0$ in Proposition (3.2). Then, C_0, ϵ_0, τ exist so that

$$N(0,\tau) \leqslant C_0 N(0,0) \leqslant \epsilon_0\sqrt{1+C_0^2} \leqslant E_2(\overline{h}_0).$$

Hence, Proposition (3.3) applies with $T = \tau$, providing us with the constants ϵ_1 and C_1 so that

$$N(0,\tau) \leqslant C_0 N(0,0) \leqslant \epsilon_1.$$

This proves the base case.

Inductively, suppose that $N(0, n\tau) \leqslant \epsilon_1$ for $n \in \mathbb{N}$. We wish to apply Proposition (3.3) with $T = n\tau$. Note that

$$N(n\tau, n\tau) \leqslant N(0, n\tau) \leqslant \epsilon_1 \leqslant \epsilon_0 \leqslant \epsilon_0\sqrt{1 + C_0^2} \leqslant E_2(\overline{h}_0).$$

Since everything is still sufficiently small (by assumption), we can apply Proposition (3.3) to conclude that

$$N(0, n\tau) \leqslant C_1 N(0,0).$$

By inductive hypothesis, we can now apply Proposition (3.2) with $T = n\tau$, which yields that $(h, \mathbf{u})$ as a doubly-periodic solution up to time $(n+1)\tau$ so that

$$N(n\tau, (n+1)\tau) \leqslant C_0 N(n\tau, n\tau).$$

Now, we have that

$$N^2(0, (n+1)\tau) \leqslant N^2(0, n\tau) + N^2(n\tau, (n+1)\tau) \leqslant (1 + C_0^2) N^2(0, n\tau).$$

In particular,

$$N(0, (n+1), \tau) \leqslant \sqrt{1 + C_0^2} N(0, n\tau) \leqslant C_1\sqrt{1 + C_0^2} N(0,0) \leqslant \epsilon_1,$$

which proves the result by induction.

Taking the limit as $n \to \infty$ returns that $N(0, \infty) \leqslant \epsilon_1$. Hence, the energy norm is bounded for all time by ϵ_1. Applying Proposition (A.3) yields that

$$\lim_{t\to\infty} \|h(t)\|_5 = 0$$

and

$$\lim_{t\to\infty} \|\mathbf{u}(t)\|_2 = 0.$$

The asymptotic convergence now follows for h from the Poincaré Inequality and the embedding $H^5(\Omega) \hookrightarrow C^{3,\alpha}(\overline{\Omega})$ and for $\mathbf{u}$ from the embedding $H^2(\Omega) \hookrightarrow C^{\alpha}(\overline{\Omega})$. $\square$

4
Conclusions and Future Work

We have shown that, given the local existence of a solution to the biharmonic SWE, we can extend the interval of existence infinitely via a priori estimates on the growth of the energy norm, allowing us to conclude that our solution pair converges to its steady-state. This helps provide justification for numerical simulations of the biharmonic shallow water equations under appropriately-small initial data and forcing considerations.

There is a myriad of potential future work. Completing the local existence argument is the most immediate follow-up. Another clear extension to this work is to prove the analogous result for the operator $\nabla^2(h\nabla^2\mathbf{u})$, some of which has been included in this paper. One could also study domain generalizations, in which case boundary estimates, such as those contained in [6] and [7], are required. This would also allow considerations of more interesting boundary conditions. Forcing generalizations are also viable extensions. One could study if a similar result could be obtained exclusively through appropriate control of the viscosity parameter (with reasonable data assumptions). We neglected to drop the viscosity parameter through many lemmas for this particular purpose. A final extension to our work is to investigate a similar result using the α-model.

Appendix A

Appendix

We need an appropriate elliptic estimate for the proof of Lemma (3.3). This proposition is a modification of Theroem 8.3 in [8]. Here, we see a significant benefit of working with doubly-periodic functions defined over L^2-Sobolev spaces; Fourier analysis becomes a viable mechanism. Noting that ∇^2 clearly satisfies the ellipticity condition, we have the following proposition.

Proposition A.1. *For all $k \geqslant 0$,*

$$\|D\mathbf{u}\|_{k+1} \leqslant \left\|\nabla^2 \mathbf{u}\right\|_k$$

for all $\mathbf{u} \in H^{k+2}(\mathbb{T}^n)$.

Proof of Proposition (A.1)*.* Without loss of generality, we will only consider the case when $\mathbf{u} \in H^2(\mathbb{T}^n)$ (otherwise, we will follow a similar process, with application of the binomial theorem). Letting $\mathbf{k} = (k_1, k_2)$, we can write

$$\mathbf{u} = \sum_{k_1} \sum_{k_2} c_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},$$

where $c_{\mathbf{k}}$ denote the Fourier coefficients

$$c_{\mathbf{k}} = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \mathbf{u}(\mathbf{x}') e^{-i\mathbf{k}\cdot\mathbf{x}'} d\mathbf{x}'.$$

Differentiating,

$$D_j \mathbf{u} = \sum_{k_1} \sum_{k_2} c_{\mathbf{k}} k_j e^{i\mathbf{k}\cdot\mathbf{x}},$$

so and $\nabla^2 \mathbf{u}$ will be given by

$$\nabla^2 \mathbf{u} = \sum_{k_1} \sum_{k_2} c_{\mathbf{k}} |\mathbf{k}|^2 e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Now, using equivalent norms,

$$\begin{aligned}
\|D\mathbf{u}\|_1^2 &= (2\pi)^n \sum_{k_1} \sum_{k_2} |c_{\mathbf{k}}|(k_1 + k_2 + k_1^2 + k_2^2) \leqslant (2\pi)^n \sum_{k_1} \sum_{k_2} |c_{\mathbf{k}}|(k_1 + k_2 + k_1^2 + k_2^2 + 2k_1 k_2) \\
&\leqslant C_1 \sum_{k_1} \sum_{k_2} |c_{\mathbf{k}}|(k_1 + k_2)^2 \leqslant C_2 \sum_{k_1} \sum_{k_2} |c_{\mathbf{k}}|(k_1^2 + k_2^2) \\
&= C_2 \left\|\nabla^2 \mathbf{u}\right\|_{L^2(\Omega)}^2,
\end{aligned}$$

where $C_1, C_2 \in \mathbb{R}^+$ are constants, by using the simple fact that $(x+y)^2 \leqslant K(x^2+y^2)$ (in fact, $K = 2$.) For higher k, one utilizes the same process, with the addition of an application of the binomial theorem, which we omit. □

Corollary A.1. *Define* $\mathbf{u}$ *as in* (3.1)*. Then,*

$$\int_0^t \|D\mathbf{u}\|_7 \, d\tau \leqslant \int_0^t C \left\|\nabla^2 \mathbf{u}\right\|_6^2 d\tau$$

some $C \in \mathbb{R}^+$.

Proof. Merely note that $\Omega = (0,1) \times (0,1) \cong \mathbb{T}^2$, apply Proposition (A.1) with $k = 6$, then integrate in time. Note that this is true for all k from 0 through 6. □

In order to encourage future work with the operator $\nabla^2(h\nabla^2 \mathbf{u})$, we provide an important tool in estimating the right-hand side of the perturbed momentum equations, as well as an example estimate. Let h be defined as in (3.1), with the constraint (3.6) given in Proposition (3.3). Then, we can obtain the following estimate.

Proposition A.2. *Let $\tilde{h} = h + \hat{h}$ be defined as described above. Then, for $k = 4$,*

$$\left\|\tilde{h}^{-1}\right\|_k^2 \leqslant C \sum_{l=0}^{k} \left\|D\tilde{h}\right\|_k^{2l},$$

where C is a time-independent constant.

Recall that the notation $\tilde{h}^{-1}$ is defined to be $\frac{1}{\tilde{h}}$.

Proof of Proposition (A.2). We will only prove this for $k = 0, 1, 2$, since all cases after $k = 2$ contain similar details. First, let $k = 0$. Then,

$$\left\|\tilde{h}^{-1}\right\|_{L^2(\Omega)}^2 = \int_\Omega (\tilde{h}^{-1})^2 \, d\mathbf{x} \leqslant C,$$

using (3.5).

Next, let $k = 1$. Then, again using (3.5),

$$\begin{aligned}
\left\|\tilde{h}^{-1}\right\|_1^2 &= \int_\Omega (\tilde{h}^{-1})^2 \, d\mathbf{x} + \int_\Omega (D(\tilde{h}^{-1}))^2 \, d\mathbf{x} \\
&\leqslant C + \int_\Omega \left(\frac{D\tilde{h}}{\tilde{h}}\right)^2 \, d\mathbf{x} \leqslant C(1 + \int_\Omega (D\tilde{h})^2 \, d\mathbf{x} \\
&= C\left(1 + \left\|D\tilde{h}\right\|_{L^2(\Omega)}^2\right) \leqslant C\left(1 + \left\|D\tilde{h}\right\|_1^2\right) = C\sum_{l=0}^{1} \left\|D\tilde{h}\right\|_k^{2l}.
\end{aligned}$$

Let $k = 2$. Once again, using (3.5), we can establish that

$$\begin{aligned}
\left\|\tilde{h}^{-1}\right\|_2^2 &= \int_\Omega (\tilde{h}^{-1})^2 \, d\mathbf{x} + \int_\Omega (D(\tilde{h}^{-1}))^2 \, d\mathbf{x} + \int_\Omega (D^2(\tilde{h}^{-1}))^2 \, d\mathbf{x} \\
&\leqslant C + C\left\|D\tilde{h}\right\|_{L^2(\Omega)}^2 + \int_\Omega \left(\frac{\tilde{h}D^2\tilde{h} - (D\tilde{h})^2}{\tilde{h}^2}\right)^2 \, d\mathbf{x} \\
&\leqslant C + C\left\|D\tilde{h}\right\|_2^2 + \int_\Omega \left(\frac{D^2\tilde{h}}{\tilde{h}}\right)^2 \, d\mathbf{x} + \int_\Omega \left(\frac{D\tilde{h}}{\tilde{h}}\right)^4 \, d\mathbf{x} + \left|2\int_\Omega \frac{D^2\tilde{h}(D\tilde{h})^2}{\tilde{h}^3} \, d\mathbf{x}\right|
\end{aligned}$$

$$= C + C\left\|D\tilde{h}\right\|_2^2 + \left\|D^2\tilde{h}\right\|_{L^2(\Omega)}^2 + \int\limits_\Omega \left(\frac{D\tilde{h}}{\tilde{h}}\right)^4 \, d\mathbf{x} + \left|2\int\limits_\Omega \frac{D^2\tilde{h}(D\tilde{h})^2}{\tilde{h}^3} \, d\mathbf{x}\right|.$$

Call these last two integral terms I_1 and I_2, respectively (from left to right). We will bound them separately, then apply them to the above.

Using that $H^2(\Omega)$ is a Banach Algebra (see [1], Theorem 5.23) and (3.5), we have that

$$I_1 = \int\limits_\Omega \left(\frac{D\tilde{h}}{\tilde{h}}\right)^4 \, d\mathbf{x} = \int\limits_\Omega \left(\left(\frac{D\tilde{h}}{\tilde{h}}\right)^2\right)^2 \, d\mathbf{x} \leqslant C\int\limits_\Omega \left(\left(D\tilde{h}\right)^2\right)^2 = C\left\|D\tilde{h}D\tilde{h}\right\|_{L^2(\Omega)}^2$$
$$\leqslant C\left\|D\tilde{h}D\tilde{h}\right\|_2^2 \leqslant C\left\|D\tilde{h}\right\|_2^2\left\|D\tilde{h}\right\|_2^2 = C\left\|D\tilde{h}\right\|_2^4,$$

where C is a time-independent constant on the parameters coming from the Banach algebra norm property on $H^2(\Omega)$.

For I_2, note that, using (3.5), Young's Inequality (with, say $\epsilon = 1$), Hölder's Inequality, and the Banach Algebra Property of $H^2(\Omega)$ (in that order),

$$I_2 = \left|2\int\limits_\Omega \frac{D^2\tilde{h}(D\tilde{h})^2}{\tilde{h}^3} \, d\mathbf{x}\right| \leqslant C\sup_{\mathbf{x}\in\Omega}|(D\tilde{h})^2|\int\limits_\Omega |D^2\tilde{h}| \, d\mathbf{x} \leqslant C\left\|(D\tilde{h})^2\right\|_2\left\|D^2\tilde{h}\right\|_{L^1(\Omega)}$$
$$\leqslant C\left(\left\|(D\tilde{h})^2\right\|_2^2 + \left\|D^2\tilde{h}\right\|_{L^1(\Omega)}^2\right) \leqslant C\left(\left\|D\tilde{h}\right\|_2^4 + \left\|D^2\tilde{h}\right\|_{L^2(\Omega)}^2\right)$$
$$\leqslant C\left(\left\|D\tilde{h}\right\|_2^4 + \left\|D\tilde{h}\right\|_2^2\right),$$

where C is a time-independent constant depending on our choice of ϵ and the Banach algebra norm property.

Plugging in our estimates,

$$\left\|\tilde{h}^{-1}\right\|_2^2 \leqslant C + C\left\|D\tilde{h}\right\|_2^2 + \left\|D^2\tilde{h}\right\|_{L^2(\Omega)}^2 + C\left\|D\tilde{h}\right\|_2^4 + C\left(\left\|D\tilde{h}\right\|_2^2 + \left\|D\tilde{h}\right\|_2^4\right)$$
$$\leqslant C\left(1 + \left\|D\tilde{h}\right\|_2^2 + \left\|D\tilde{h}\right\|_2^4\right) = C\sum_{l=0}^{2}\left\|D\tilde{h}\right\|_2^{2l}.$$

Following a similar process for $k = 3, 4$ provides the desired result. □

Our example estimate will demonstrate how to utilize the above to obtain a bound on the $\mathbf{R}$ from (3.2) up to, say, the H^3-norm:

Lemma A.1.

$$\sup_{\tau\in[0,t]} \|\mathbf{R}\|_3^2 \leqslant CN^2(0,t)\sum_{l=1}^{4}\left(N^{2l}(0,t) + \|DV\|_5^{2l}\right),$$

where C is a time-independent constant.

Proof of Lemma (A.1). Using the Banach algebra norm property of $H^3(\Omega)$, Proposition (A.2), and the triangle inequality, we compute that

$$\begin{aligned}
\sup_{\tau\in[0,t]} \|\mathbf{R}\|_3^2 &= \sup_{\tau\in[0,t]}\left\{\left\|(\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{\lambda}{h+\hat{h}}\nabla^2(h+\hat{h})\nabla^2\mathbf{u} - \frac{2\lambda}{h+\hat{h}}\nabla^3\mathbf{u}\nabla(h+\hat{h})\right\|_3^2\right\}\\
&\leqslant C\sup_{\tau\in[0,t]}\left\{\|\mathbf{u}\|_6^2\|D\mathbf{u}\|_3^2 + \lambda\left\|\nabla^2(h+\hat{h})\right\|_3^2\left\|(h+\hat{h})^{-1}\right\|_3^2\left\|\nabla^2\mathbf{u}\right\|_3^2\right.\\
&\left.+ 2\lambda\left\|\nabla^3\mathbf{u}\right\|_3^2\left\|\nabla(h+\hat{h})\right\|_3^2\left\|(h+\hat{h})^{-1}\right\|_3^2\right\}\\
&\leqslant C\sup_{\tau\in[0,t]}\left\{\left\|\nabla^3\mathbf{u}\right\|_3^2\left(\|\mathbf{u}\|_6^2 + \lambda\left\|\nabla^2(h+\hat{h})\right\|_3^2\left\|(h+\hat{h})^{-1}\right\|_3^2\right.\right.\\
&\left.\left.+ 2\lambda\left\|\nabla^3\mathbf{u}\right\|_3^2\left\|\nabla(h+\hat{h})\right\|_3^2\left\|(h+\hat{h})^{-1}\right\|_3^2\right)\right\}\\
&\leqslant CN^2(0,t)\sup_{\tau\in[0,t]}\left\{\|\mathbf{u}\|_6^2 + \lambda\left\|\nabla^2(h+\hat{h})\right\|_3^2\sum_{l=0}^{3}\left\|D(h+\hat{h})\right\|_3^{2l}\right.\\
&\left.+ 2\lambda\left\|\nabla(h+\hat{h})\right\|_3^2\sum_{l=0}^{3}\left\|D(h+\hat{h})\right\|_3^{2l}\right\}\\
&\leqslant CN^2(0,t)\sup_{\tau\in[0,t]}\left\{\|\mathbf{u}\|_6^2 + \lambda\left\|\nabla^2(h+\hat{h})\right\|_3^2\sum_{l=0}^{3}\left\|D^2(h+\hat{h})\right\|_3^{2l}\right.\\
&\left.+ 2\lambda\left\|\nabla(h+\hat{h})\right\|_3^2\sum_{l=0}^{3}\left\|D(h+\hat{h})\right\|_3^{2l}\right\}\\
&\leqslant CN^2(0,t)\sup_{\tau\in[0,t]}\left\{\|\mathbf{u}\|_6^2 + \lambda\sum_{l=1}^{4}\left\|D^2(h+\hat{h})\right\|_3^{2l} + 2\lambda\sum_{l=1}^{4}\left\|D(h+\hat{h})\right\|_3^{2l}\right\}\\
&\leqslant CN^2(0,t)\sup_{\tau\in[0,t]}\left\{\|\mathbf{u}\|_6^2 + \lambda\sum_{l=1}^{4}\left\|D^2h\right\|_3^{2l} + \lambda\sum_{l=1}^{4}\left\|D^2\hat{h}\right\|_3^{2l}\right.
\end{aligned}$$

$$
\begin{aligned}
&+ 2\lambda \sum_{l=1}^{4} \|Dh\|_3^{2l} + 2\lambda \sum_{l=1}^{4} \left\|D\hat{h}\right\|_3^{2l} \Big\} \\
&\leqslant CN^2(0,t)\Bigg(\lambda \sum_{l=1}^{4} N^{2l}(0,t) + \lambda \sum_{l=1}^{4} \left\|D^2V\right\|_3^{2l} \\
&+ 2\lambda \sum_{l=1}^{4} N^{2l}(0,t) + 2\lambda \sum_{l=1}^{4} \|DV\|_3^{2l}\Bigg) \\
&\leqslant CN^2(0,t) \sum_{l=1}^{4} \left(N^{2l}(0,t) + \|DV\|_5^{2l}\right),
\end{aligned}
$$

where C is a time-independent constant depending on the parameters present in the conditions of the Banach algebra norm property of $H^3(\Omega)$ and λ. which proves the result. □

In the proof of the main theorem, we require a result from elementary real analysis.

Proposition A.3. *Let f be a differentiable function on $[0,\infty)$ such that $\|f'(t)\|_{L^2([0,\infty))} < \infty$. Then,* $\lim_{t\to\infty} |f(t)| = 0$.

Proof of Proposition (A.3). Let (t_n) be a real-valued sequence so that $t_n \to \infty$ as $n \to \infty$. Then, applying the Fundamental Theorem of Calculus and the Cauchy-Schwarz Inequality,

$$
\begin{aligned}
f^2(t_m) - f^2(t_n) &= \int_{t_n}^{t_m} (f^2(\tau))' d\tau \\
&\leqslant \int_{t_n}^{t_m} 2|f(\tau)||f'(\tau)| d\tau \\
&\leqslant 2 \left(\int_{t_n}^{t_m} |f(\tau)|^2 d\tau \right)^{1/2} \left(\int_{t_n}^{t_m} |f'(\tau)|^2 d\tau \right)^{1/2} \\
&\leqslant 2 \left(\int_{t_n}^{\infty} |f(\tau)|^2 d\tau \right)^{1/2} \left(\int_{t_n}^{\infty} |f'(\tau)|^2 d\tau \right)^{1/2},
\end{aligned}
$$

for $m > n$. As $n \to \infty$, $t_n \to \infty$, so the right-hand goes to zero. This proves that the sequence $(f(x_n))$ is Cauchy, and it can be shown that it has a limit of 0. Now, the result follows directly from the sequential characterization of limits. □

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