AN INTRODUCTION TO SEMIGROUP THEORY

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Abstract. In this project, we provide an introduction to semigroup theory, which we then apply to parabolic PDEs. We start by discussing elementary properties of semigroups and their infinitesimal generator. From there, we proceed onto related resolvent properties and end our semigroup theory with a proof of the Hille-Yosida Theorem, along with statements of some consequences. Finally, we consider a homogeneous parabolic problem, and apply introduced semigroup theory generate a unique solution.

Contents

1. Introduction to Semigroups

1.1. Motivation. First, let $X = \mathbb{R}^n$, and consider the ODE initial-value problem

$$\begin{cases} u'(t) = Au(t) \\ u(0) = u_0, \end{cases}$$

where A is a linear operator on X. Since X is finite dimensional, A is bounded, and we know that the unique solution to the above ODE is

$$u(t) = e^{tA}u_0.$$

One can view $e^{tA} : \mathbb{R}^n \to \mathbb{R}^n$ as the map $u_0 \mapsto e^{tA}u_0$, which is a map of initial conditions along trajectories of the ODE. This is called the *flow* of the system. If we denote $S(t) = e^{tA}$, then we note that $S(t)$ has the following elementary properties for any $x \in \mathbb{R}^n$:

- $S(0)x = x$
- $S(s)(S(t)x) = S(s+t)x$ for all $s, t \in \mathbb{R}$
- $\|S(t)x - x\| \to 0$ as $t \to 0$
- The stable, unstable, and center subspaces are all invariant under $S(t)$.

A natural question to ask is what happens if X is a (real) Banach space of infinite dimension. Now, A need not be bounded. If A is bounded, then we can use the Riesz functional calculus to define e^{tA} as a power series, and everything works out as before. If not, we must specify a subspace $D(A) \subset X$, then we get an unbounded operator $A : D(A) \to X$. Since we cannot use the same functional calculus (although there is more general function calculus such as the *continuous* or *Borel* functional calculus), it is not clear how to extend our previous notions. So, we must ask, if A is unbounded, can we obtain a unique solution $u : [0, \infty) \to X$, for any initial condition? Many parabolic and hyperbolic PDEs can be put into an ODE form, so this is certainly a problem of interest.

1.2. **Semigroups.** Let X be a real Banach space. In parallel to the properties of the flow in the finite-dimensional problem, we define a semigroup as follows.

Definition 1. A family $\{S(t)\}_{t\geq 0}$ of bounded linear operators on X is called a (one-parameter) semigroup if

- $S(0)u = u$ for all $u \in X$
- $S(t+s)u = S(t)S(s)u = S(s)S(t)u$ for all $s, t \in [0, \infty),\ u \in X$
- For each $u \in X$, the map $t \mapsto S(t)u$ is continuous from $[0, \infty)$ into X,

Note that $S : [0, \infty) \to \mathcal{L}(X)$ (the space of bounded, linear operators on X), and the first two conditions indicate that S is a representation for the additive group of non-negative reals, whereas the last is a topological concern on the semigroup elements. Writing the last condition differently, we have that as $t \to 0$, $S(t)u \to u$ in the X norm $\|\cdot\|$, so this means that S is continuous in the strong operator topology (such a semigroup is sometimes referred to as a *strongly continuous* semigroup).

If a semigroup has the property that $\|S(t)\|_{op} \leq 1$ for all $t \geq 0$, then the family is called a *contraction semigroup*. This will be our focus.

Let $\{S(t)\}_{t\geq 0}$ be a contraction semigroup, and

$$D(A) = \left\{ u \in X : \lim_{t\to 0^+} \frac{S(t)u - u}{t} \text{ exists in } X \right\}.$$

Then, we can define an (often) unbounded operator $A : D(A) \to X$ by

$$Au = \lim_{t\to 0^+} \frac{S(t)u - u}{t},$$

for $u \in D(A)$. A is called the *infinitesimal generator* of the semigroup.

Example. Consider the standard heat equation on $\mathbb{R}^n_x \times \mathbb{R}^+_t$, with $u = f$ on $\mathbb{R}^n \times \{t = 0\}$. We saw in class that the solution is given by

$$u(x,t) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y)\ dy.$$

Denote this by $S(t)f(x)$. That is, $u(x,t) = S(t)f(x) = (k_t *_x f)(x,t)$, where k_t is the heat kernel. As discussed in class, $S(0) = I$, and one can check that both $S(t+s) = S(t)S(s)$ and $\lim_{t\to 0} S(t)f = f$. Additionally, $S(t)$ is defined on $L^p(\mathbb{R}^n)$, with $\|S(t)\|_{op} \leq 1$. Hence, $\{S(t)\}_{t\geq 0}$ is a contraction semigroup on $L^p(\mathbb{R}^n)$. It can be shown, although we will not discuss the details, that the generator of this semigroup is Δ, with $D(\Delta) = H^2(\mathbb{R}^n)$. This is sometimes referred to as the *heat semigroup*.

Here are some elementary facts about semigroups and their generators:

Proposition 1.1. *Let $u \in D(A)$.*

(1) $S(t)D(A) \subset D(A)$

(2) *$AS(t)u = S(t)Au$ for any $t > 0$*
(3) *The map $t \mapsto S(t)u$ is differentiable for each $t > 0$, with*

$$\frac{d}{dt}S(t)u = AS(t)u.$$

(4) *A is a closed, densely-defined operator.*

The second and third parts of the above motivate the notation $S(t) = e^{tA}$ which is used in certain cases, such as with $A = \Delta$. In certain instances, like with $A = \Delta$, this can be made rigorous using Borel functional calculus for A, say, self-adjoint. The first three are direct, so we omit their proofs. The fourth part is more interesting.

Proof of (4). First, we show density. Let $u \in X$, and define the sequence

$$u^t = \int_0^t S(s)u \, ds.$$

Since the mapping $s \mapsto S(s)u$ is continuous, we have

$$\left\| \frac{u^t}{t} - u \right\| = \left\| \frac{1}{t} \int_0^t (S(s)u - u) \, ds \right\| \leq \frac{1}{t} \int_0^t \|S(s)u - u\| \, ds \to 0,$$

as $t \to 0^+$. This shows that $u^t/t \to u$ in X. It remains to show that $u^t/t \in D(A)$. Clearly, it suffices to prove that $u^t \in D(A)$. This is not too difficult:

$$\begin{aligned}
\frac{S(h)u^t - u^t}{h} &= \frac{1}{h}\left(S(h) \int_0^t S(s)u \, ds - \int_0^t S(s)u \, ds \right) = \frac{1}{h}\left(\int_0^t S(s+h)u \, ds - \int_0^t S(s)u \, ds \right) \\
&= \frac{1}{h}\left(\int_h^{t+h} S(s)u \, ds - \int_0^t S(s)u \, ds \right) = \frac{1}{h}\left(\int_t^{t+h} S(s)u \, ds - \int_0^h S(s)u \, ds \right) \\
&\to S(t)u - S(0)u = S(t)u - u
\end{aligned}$$

as $h \to 0^+$, via continuity of the semigroup. Thus, $u^t \in D(A)$ (and, in particular, $Au^t = S(t)u - u$).

Next, we show that the operator is closed. Let $u_j \in D(A)$ converge to u, and $Au_j \to v$. Well,

$$Au_j = \lim_{t \to 0^+} \frac{S(t)u_j - u_j}{t},$$

and by the fundamental theorem of calculus,

$$S(t)u_j - u_j = \int_0^t S(s) \; Au_j \, ds,$$

since

$$\frac{d}{dt}S(t)u = AS(t)u = S(t)Au.$$

Taking the limit as $j \to \infty$, we have that $S(t)u - u = \int_0^t S(s)v\, ds$, and so

$$\lim_{t\to 0^+} \frac{S(t)u - u}{t} = \lim_{t\to 0^+} \frac{1}{t} \int_0^t S(s)v\, ds = v,$$

and so $u \in D(A)$ by the definition of $D(A)$, and $v = Au$ by the definition of A. □

1.3. **Resolvents and the Hille-Yosida Theorem.** First, we recall the definition of the resolvent of a closed, densely-defined operator $A : X \to X$. We say $\zeta \in \mathbb{C}$ belongs to the resolvent set $\rho(A)$ if and only if $\zeta I - A : D(A) \to X$ is bijective (we will now write $\zeta - A$). For such a ζ, the inverse map $R_\zeta = (\zeta - A)^{-1} : X \to D(A) \subset X$ is called the resolvent of A. The complement of the resolvent in $\mathbb{C}$, denoted $\sigma(A)$, is called the spectrum of A. By the closed graph theorem, R_ζ is bounded if and only if its graph is closed, and its graph is the flipped graph of $\zeta - A$, which is closed if and only if A is closed. In particular, $R_\zeta \in \mathcal{L}(X)$. Further, A and R_ζ commute, which will be shown in the proof of the upcoming theorem. Here are a couple of important properties of the resolvent, including an interpretation of its action as being that of the Laplace transform of the semigroup.

Theorem 1.2. (1) *If $\lambda, \mu \in \rho(A)$, then*

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$$

and

$$R_\lambda R_\mu = R_\mu R_\lambda.$$

(2) *If $\lambda > 0$, then $\lambda \in \rho(A)$,*

$$R_\lambda u = \int_0^\infty e^{-\lambda t} S(t)u\, dt$$

for $u \in X$, and $\|R_\lambda\|_{op} \le \frac{1}{\lambda}$.

Proof. (1) Note that for any $\lambda \in \rho(A)$, R_λ commutes with $\lambda - A$, so it commutes with A, and so it commutes with any $\mu - A$. Further, if $\mu \in \rho(A)$, then

$$R_\lambda R_\mu = R_\lambda R_\mu (\lambda - A) R_\lambda = R_\lambda (\lambda - A) R_\mu R_\lambda = R_\mu R_\lambda.$$

Hence,

$$R_\lambda R_\mu = R_\mu R_\lambda,$$

and so

$$R_\lambda - R_\mu = (\mu - A) R_\lambda R_\mu - (\lambda - A) R_\mu R_\lambda = (\mu - \lambda) R_\lambda R_\mu.$$

(2) Let us call the right-hand side $\tilde{R}_\lambda u$. Clearly, $\tilde{R}_\lambda \in \mathcal{L}(X)$, since $\{S(t)\}_{t\geq 0}$ is a contraction semigroup. First, if $u \in D(A)$, then

$$\begin{aligned}
\tilde{R}_\lambda(\lambda - A)u &= \int_0^\infty e^{-\lambda t} S(t)(\lambda - A)u \, dt \\
&= \int_0^\infty \lambda e^{-\lambda t} S(t)u \, dt - \int_0^\infty e^{-\lambda t} S(t)Au \, dt \\
&= \int_0^\infty \lambda e^{-\lambda t} S(t)u \, dt - \int_0^\infty e^{-\lambda t} \frac{d}{dt} S(t)u \, dt \\
&= \int_0^\infty \lambda e^{-\lambda t} S(t)u \, dt - e^{-\lambda t} S(t)u\big|_0^\infty - \lambda \int_0^\infty e^{-\lambda t} S(t)u \, dt \\
&= u.
\end{aligned}$$

That is, $\tilde{R}_\lambda(\lambda - A)u = u$ on $D(A)$. Next, we show that $\tilde{R}_\lambda : X \to D(A)$, and $(\lambda - A)\tilde{R}_\lambda u = u$, for any $u \in X$ which will complete the proof. Indeed, for any $h > 0$ and $u \in X$,

$$\begin{aligned}
\frac{S(h)\tilde{R}_\lambda u - \tilde{R}_\lambda u}{h} &= \frac{1}{h} \int_0^\infty e^{-\lambda t} \left(S(t+h)u - S(t)u \right) \, dt \\
&= \frac{1}{h} \left(\int_h^\infty e^{-\lambda(t-h)} S(t)u \, dt - \int_0^\infty e^{-\lambda t} S(t)u \, dt \right) \\
&= \frac{1}{h} \left(e^{\lambda h} \left(\int_0^\infty e^{-\lambda t} S(t)u \, dt - \int_0^h e^{-\lambda t} S(t)u \, dt \right) - \int_0^\infty e^{-\lambda t} S(t)u \, dt \right) \\
&= \frac{e^{\lambda h} - 1}{h} \tilde{R}_\lambda u - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} S(t)u \, dt \\
&\to \lambda \tilde{R}_\lambda u - u,
\end{aligned}$$

as $h \to 0^+$. (again, using strong continuity). Thus, $\tilde{R}_\lambda u \in D(A)$, and by definition of A,

$$A\tilde{R}_\lambda u = \lambda \tilde{R}_\lambda u - u,$$

or

$$(\lambda - A)\tilde{R}_\lambda u = u.$$

Thus, $\tilde{R}_\lambda = R_\lambda$. The bound on the operator norm is obvious.

□

Armed with this, we show that the infinitesimal generator uniquely defines the semigroup.

Proposition 1.3. *If $\{S(t)\}_{t\geq 0}$ and $\{P(t)\}_{t\geq 0}$ are two one-parameter contraction semigroups with the same infinitesimal generator, then $S(t) = P(t)$ for all $t \in [0,\infty)$.*

Proof. If $\lambda > 0$, then for any $u \in X$ and $u' \in X'$,

$$\int_0^\infty e^{-\lambda t}\langle u', S(t)u\rangle \; dt = \langle u', R_\lambda u\rangle = \int_0^\infty e^{-\lambda t}\langle u', P(t)u\rangle \; dt.$$

Now, the Laplace transform for real-valued functions is unique, so $\langle u', S(t)u\rangle = \langle u', P(t)u\rangle$. Since this is true for any $t \in [0,\infty)$, $u \in X$, and $u' \in X'$, the Hahn-Banach theorem implies that $S(t)u = P(t)u$ for all $u \in X$.

Alternatively, note that if $u \in D(A)$, then

$$\frac{d}{ds}\left(S(s)P(t-s)u\right) = 0,$$

and so $S(s)P(t-s)u$ is constant in s, for any t. Evaluating at $s = 0$ and $s = t$, we get that $S(t)u = P(t)u$ for all t, and $u \in D(A)$. Since $D(A)$ is dense in X, this extends to all $u \in X$. ☐

Now, we can fully describe the infinitesimal generator.

Theorem 1.4 (Hille-Yosida Theorem)**.** *Let A be a closed, densely-defined linear operator X. Then, A is the generator of a contraction semigroup $\{S(t)\}_{t\geq 0}$ if and only if*

$$(0,\infty) \subset \rho(A) \qquad \text{and } \left\|R_\lambda\right\|_{op} \leq \frac{1}{\lambda}$$

for $\lambda > 0$.

Proof. ($\Longrightarrow$) This follows immediately From Theorem 1.2 (2).

($\Longleftarrow$) We must construct the semigroup. Let $\lambda > 0$, and define the *Yosida approximation* of A: $A_\lambda = -\lambda + \lambda^2 R_\lambda$. Noting that by the definition of R_λ,

$$-\lambda + \lambda^2 R_\lambda = \lambda(-I + \lambda R_\lambda) = \lambda(-(\lambda - A)R_\lambda + \lambda R_\lambda) = \lambda A R_\lambda,$$

so $A_\lambda = \lambda A R_\lambda$. First, we claim that $A_\lambda \to A$ as $\lambda \to \infty$ in the strong operator topology on $D(A)$.

Note that dividing through by λ yields $\lambda R_\lambda u - u = AR_\lambda u = R_\lambda A u$, and so by the operator norm inequality,

$$\|\lambda R_\lambda u - u\| = \|R_\lambda A u\| \leq \|R_\lambda\|_{op}\|Au\| \leq \frac{1}{\lambda}\|Au\|.$$

Taking the limit as $\lambda \to \infty$, we get that $\lambda R_\lambda \to I$ in the strong operator topology on $D(A)$. Keeping in mind that $\|\lambda R_\lambda\| \leq 1$ and $D(A)$ is dense in X, this extends to convergence on X. Indeed, fix $\epsilon > 0$ and $u \in X$. By density, there exists $v \in D(A)$ so that $\|u - v\| < \epsilon/3$, since $v \in D(A)$, there exists λ_0 so that $\|\lambda R_\lambda v - v\| < \epsilon/3$ for all $\lambda \geq \lambda_0$. Then, for any $\lambda \geq \lambda_0$

$$\begin{aligned}
\|\lambda R_\lambda u - u\| &\leq \|\lambda R_\lambda u - \lambda R_\lambda v\| + \|\lambda R_\lambda v - v\| + \|v - u\| \\
&\leq \|\lambda R_\lambda\|_{op}\|v - u\| + \|\lambda R_\lambda v - v\| + \|v - u\| \\
&\leq 2\|u - v\| + \|\lambda R_\lambda v - v\| \\
&< \epsilon.
\end{aligned}$$

Since for any $u \in D(A)$, we have $Au \in X$, and then $A_\lambda u = \lambda A R_\lambda u = \lambda R_\lambda A u$, the above work establishes the claim. So, we have a sequence approximating A, in the appropriate sense. Now, we will define our semigroup in the same way.

Let

$$S_\lambda(t) = e^{tA_\lambda} = e^{-\lambda t} e^{\lambda^2 t R_\lambda} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda^2 t)^k}{k!} R_\lambda^k,$$

where we have used the functional calculus in the last line (R_λ is bounded). Clearly, this is a semigroup (by properties of exponential functions, which hold in the functional calculus). In fact, since the space of bounded linear operators with the operator norm is a Banach algebra,

$$\|S_\lambda(t)\|_{op} \le e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda^2 t)^k}{k!} \|R_\lambda\|_{op}^k \le e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} = e^{-\lambda t} e^{\lambda t} = 1,$$

using that $\|R_\lambda\| \le \lambda^{-1}$. Hence, $\{S_\lambda(t)\}_{t\ge0}$ constitutes a contraction semigroup. It is not difficult to see that it has infinitesimal generator A_λ, on $D(A_\lambda) = X$. We would like to show that $S_\lambda t$ converges as $\lambda \to \infty$ in the strong operator topology, for every t. To do this, we prove that it is Cauchy.

Let $\lambda, \mu > 0$. By theorem 1.2, A_λ and A_μ commute, so A_μ and $S_\lambda(t)$ commute for all $t > 0$. Using this, we obtain that for all $u \in D(A)$,

$$\begin{aligned} S_\lambda(t)u - S_\mu(t)u &= \int_0^t \frac{d}{ds}\left(S_\mu(t-s)S_\lambda(s)u\right)\, ds \qquad \text{(by the FTC and the commuting property)} \\ &= \int_0^t S_\mu(t-s)S_\lambda(s)(A_\lambda - A_\mu)u\, ds \qquad \text{(by Prop 1.1.2-3, and the product rule)}, \end{aligned}$$

and so

$$\begin{aligned} \|S_\lambda(t)u - S_\mu(t)u\| &\le \sup_s \|S_\mu(t-s)\|_{op} \sup_s \|S_\lambda(s)\|_{op} \|A_\lambda u - A_\mu u\| \int_0^t ds \\ &\le t\|A_\lambda u - A_\mu u\| \le t\left(\|A_\lambda u - Au\| + \|A_\mu u - Au\|\right) \\ &\to 0 \quad \text{ as } \lambda, \mu \to \infty \end{aligned}$$

since $\{S_\lambda(t)\}_{t\ge0}$ is a contraction semigroup and $A_\lambda u,\ A_\mu u \to u$ in the strong operator topology. Since X is a Banach space, this Cauchy sequence has a limit for all t. Define $S(t)u = \lim_{\lambda\to\infty} S_\lambda(t)u$, for all $t \ge 0,\ u \in D(A)$. Since our semigroup is contractive, this extends to all $u \in X$ and the convergence is locally uniform in t. The local unique uniform is obvious. For the extension, we proceed similar to our earlier extension for strong operator convergence: use density to approximate u by some $v \in D(A)$ in norm, then observe the string of inequalities

$$\begin{aligned} \|S_\lambda(t)u - S_\mu(t)u\| &\le \|S_\lambda(t)u - S_\lambda(t)v\| + \|S_\lambda(t)v - S_\mu(t)v\| + \|S_\mu(t)v - S_\mu(t)u\| \\ &\le \|S_\lambda\|_{op}\|u - v\| + \|S_\lambda(t)v - S_\mu(t)v\| + \|S_\mu(t)\|_{op}\|u - v\| \\ &\le 2\|u - v\| + \|S_\lambda(t)v - S_\mu(t)v\|, \end{aligned}$$

just as earlier in the proof. The first term is small by density, the second by the fact that $v \in D(A)$, and so $S(t)v = \lim_{\lambda\to\infty} S_\lambda(t)v$. This proves $(S_\lambda(t)u)$ is Cauchy in the complete space X, and so $S(t)$ extends to act on X. With all of this in hand, it is not difficult to show that $\{S(t)\}_{t\geq 0}$ is a contraction semigroup on all of X.

Finally, we show that A is indeed the generator of this semigroup. Call the generator A'. For this, we must show both the limit property and that they have the same domains. Since A_λ generates $\{S_\lambda(t)\}_{t\geq 0}$, the fundamental theorem of calculus gives

$$S_\lambda(t)u - u = \int_0^t S_\lambda(s)A_\lambda u \, ds.$$

Furthermore, if $u \in D(A)$, then

$$\begin{aligned}
\|S_\lambda(s)A_\lambda u - S(s)Au\| &= \|S_\lambda(s)A_\lambda u - S(s)Au + S_\lambda(s)Au - S_\lambda(s)Au\| \\
&\leq \|S_\lambda(s)A_\lambda u - S_\lambda(s)Au\| + \|S_\lambda(s)Au - S(s)Au\| \\
&\leq \|S_\lambda(s)\|_{op} \|A_\lambda u - Au\| + \|S_\lambda(s)Au - S(s)Au\| \\
&\to 0 \qquad \text{as } \lambda \to \infty,
\end{aligned}$$

using the strong operator convergence of A_λ to A and $S_\lambda(s)$ to $S(s)$. Applying this to our previous statement gives that

$$S(t)u - u = \int_0^t S(s)Au \, ds,$$

for $u \in D(A)$. Dividing both sides by t and applying strong continuity yields that $u \in D(A')$, and $A'u = Au$ for $u \in D(A)$. It remains to show that $D(A) = D(A')$. Let $\lambda > 0$. Then, we know $\lambda \in \rho(A) \cap \rho(A')$ (the former by assumption, the latter by Theorem 1.2). Hence, $\lambda - A$, $\lambda - A'$ send $D(A)$ to X bijectively. Since $D(A) \subseteq D(A') \subseteq X$, we must have $D(A) = D(A')$. Thus, $A = A'$, and since the infinitesimal generator is unique by Proposition 1.3, we are done. □

Remark 1.5. (1) The above shows that

$$e^{tA_\lambda}u \to S(t)u$$

strongly. Since $A_\lambda \to A$ strongly, this provides another manner to interpret the notation $S(t)u = e^{tA}u$.

(2) There is a more general version of this theorem, where one does not require that the semigroup be contractive. Here, the statement takes the following form:

A densely-defined operator A is the infinitesimal generator of a semigroup $\{S(t)\}_{t\geq 0}$ if and only if there exist constants C, γ so that

$$\|R_\lambda^m\| \leq C(\lambda - \gamma)^{-m}$$

for all $\lambda > \gamma$ and $m \in \mathbb{N}$. See [4] for details.

Remark 1.6. Another variant is for ω-contraction semigroups, where the semigroup satisfies the property $\|S(t)\|_{op} \leq e^{\omega t}$). In fact, this is actually a special case of the general version above (using the uniform boundedness principle). The variant is that a closed, densely-defined linear operator A generates an ω-contraction semigroup if and only if $(\omega, \infty) \subset \rho(A)$ and $\|R_\lambda\|_{op} \leq (\lambda - \omega)^{-1}$ for all $\lambda > \omega$. The proof is similar to ours, and thus omitted. However, we will use this version for our parabolic application.

Before moving on, we list an important consequence of the Hille-Yosida theorem called *Stone's Theorem*. This has application in quantum mechanics, as the Hamiltonian is the infinitesimal generator of the time evolution semigroup of the state space of an isolated quantum mechanical system.

Theorem 1.7 (Stone's Theorem). *If A is self-adjoint, then iA generates the unitary group $U(t) = e^{itA}$. Conversely, if $\{U(t)\}_{t\geq 0}$ is a semigroup of unitary operators, then there exists a self-adjoint operator A so that $U(t) = e^{itA}$.*

Here, one can rigorously define e^{itA} via the Borel functional calculus. This theorem is beneficial viewpoint for the solution operator to the Schrödinger equation (often referred to as the *Schrödinger propagator*).

2. Application of Semigroups to Parabolic PDEs

Consider the parabolic problem

$$\begin{cases} \partial_t u + Lu = 0 \text{ in } U_T \\ u = 0 \text{ on } \partial U \times [0,T] \\ u = g \text{ on } U \times \{t = 0\} \end{cases}$$

where $U \subset \mathbb{R}^n$ is open, bounded, and has smooth boundary, $U_T = U \times (0,T]$, and $T > 0$ is fixed. We assume that L is uniformly elliptic and has the form

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u,$$

where the coefficients are smooth on $\bar{U}$ and time-independent. We will recast the problem using semigroups, with $X = L^2(U)$. Set $D(A) = H_0^1(U) \cap H^2(U)$, and $Au = -Lu$, for $u \in D(A)$. So, we now are in the setting of the *abstract Cauchy problem*

$$\begin{cases} u'(t) = Au(t), \\ u(0) = g \end{cases}$$

for $u \in D(A)$.

Before moving on to solving the problem, we recall a few facts from [1]. The first is the energy estimate

$$(*) \qquad \beta \left\|u\right\|_{H_0^1(U)}^2 \leq B[u,u] + \gamma \left\|u\right\|_{L^2(U)}^2,$$

for some $\beta > 0$, $\gamma \geq 0$, where

$$B[u,v] = \int_U \sum_{i,j=1}^{n} a_{ij}u_{x_i}v_{x_j}\, dx + \int_U \sum_{i=1}^{n} b_i u_{x_i} v\, dx + \int_U cuv\, dx$$

is the bilinear form defined on $H_0^1(U)$ associated to L. Next, we will record some elliptic theory.

Theorem 2.1 (Existence Theorem). *There exists $\gamma \geq 0$ so that for all $\mu \geq \gamma$ and each $f \in L^2(U)$, there exists a unique, weak solution $u \in H_0^1(U)$ to the problem*

$$\begin{cases} Lu + \mu u = f \text{ in } U, \\ u = g \text{ on } \partial U \end{cases}$$

Theorem 2.2 (Boundary Regularity)**.** *Assume $a_{ij} \in C^1(\bar{U})$, $b_i, c \in L^\infty(U)$, and $f \in L^2(U)$. If $u \in H_0^1(U)$ is a weak solution of the elliptic boundary-value problem*

$$\begin{cases} Lu = f \text{ in } U, \\ \quad u = 0 \text{ on } \partial U \end{cases}$$

where ∂U is C^2, then $u \in H^2(U)$, and we have the estimate

$$\|u\|_{H^2(U)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)} \right),$$

where C depends only on U and the coefficients of L.

Now, we are ready to state the relevant theorem.

Theorem 2.3. *A generates a γ-contraction semigroup $\{S(t)\}_{t\geq 0}$ on $X = L^2(U)$.*

It is clear, in view of Proposition 1.1, that if the above theorem holds, then $S(t)g$ will be the unique solution (unique by Proposition 1.3).

Proof. We will appeal to the Hille-Yosida Theorem (the variant for γ-contraction semigroups, as mentioned in Remark 1.6). First, we must show that A is closed and densely-defined.

The fact that A is densely-defined is obvious. To show A is closed, let (u_j) be a sequence in $D(A)$, with $u_j \to u$, and $Au_j \to v$ in $L^2(U)$. We show $u \in D(A)$ and $Au = v$. From the regularity theorem, we know that for any $k, l \in \mathbb{N}$,

$$\|u_k - u_l\|_{H^2(U)} \leq C \left(\|A(u_k - u_l)\|_{L^2(U)} + \|u_k - u_l\|_{L^2(U)} \right),$$

which tells us that (u_j) is Cauchy in $H^2(U)$. Note that we may apply this theorem because $u_k - u_l$ clearly solves $L(u_k - u_l) = f$, where $f = -A(u_k - u_l)$, and $u_k - u_l$ has sufficient regularity by assumption. By completeness and uniqueness of limits, $u_j \to u$ in $H^2(U)$. Since $H_0^1(U)$ is closed, this shows that $u \in D(A)$. Since $u_j \to u$ in $H^2(U)$, we have that $Au_j \to u$ in $L^2(U)$, and thus $Au = v$. Hence, A is closed and densely-defined.

Next, we show that $(\gamma, \infty) \subset \rho(A)$. Fix $\lambda \geq \gamma$, and consider the problem

$$\begin{cases} Lu + \lambda u = f \text{ in } U, \\ \qquad\quad u = 0 \text{ on } \partial U \end{cases}$$

By our existence theorem, this has a unique, weak solution $u \in H_0^1(U)$, for any $f \in L^2(U)$. Elliptic regularity implies that $u \in H^2(U)$, so $u \in D(A)$. Hence, $(\lambda - A)u = f$ has a unique (weak) solution, which guarantees that $(\lambda - A) : D(A) \to X$ is bijective for any $\lambda \geq \gamma$. Thus, $[\gamma, \infty) \subset \rho(A)$.

Finally, we must show that $\|R_\lambda\|_{op} \leq (\lambda - \gamma)^{-1}$ for all $\lambda > \gamma$. The weak formulation of the previously-introduced BVP is

$$B[u, v] + \lambda(u, v)_{L^2(U)} = (f, v)_{L^2(U)},$$

valid for any $v \in H_0^1(U)$. Since $D(A) \subset H_0^1(U)$, we may take $v = u$ in the above. Using the Cauchy-Schwarz inequality,

$$B[u, u] + \lambda \|u\|_{L^2(U)} = (f, u)_{L^2(U)} \leq \|f\|_{L^2(U)} \|u\|_{L^2(U)}.$$

Write the left-hand side of the above as

$$B[u, u] + \gamma \|u\|_{L^2(U)}^2 + (\lambda - \gamma) \|u\|_{L^2(U)}^2.$$

Applying the estimate (*) yields

$$(\lambda-\gamma)\left\| u\right\|^2_{L^2(U)} \le \beta \left\| u\right\|^2_{H^1_0(U)} + (\lambda-\gamma)\left\| u\right\|^2_{L^2(U)} \le B[u,u] + \gamma \left\| u\right\|^2_{L^2(U)} + (\lambda-\gamma)\left\| u\right\|^2_{L^2(U)}.$$

Combining these results, we obtain

$$(\lambda-\gamma)\left\| u\right\|^2_{L^2(U)} \le \left\| f\right\|_{L^2(U)} \left\| u\right\|_{L^2(U)}.$$

Recall that $R_\lambda f = u$. Armed with the above estimate, we have that, for all $f \in L^2(U)$,

$$(\lambda-\gamma)\left\| R_\lambda f\right\|^2_{L^2(U)} \le \left\| f\right\|_{L^2(U)} \left\| R_\lambda f\right\|_{L^2(U)},$$

or

$$\left\| R_\lambda f\right\|_{L^2(U)} \le (\lambda-\gamma)^{-1} \left\| f\right\|_{L^2(U)}.$$

Thus,

$$\left\| R_\lambda\right\|_{op} \le (\lambda-\gamma)^{-1}.$$

This shows that A satisfies the hypotheses of the Hille-Yosida theorem for γ-contraction semigroups. Applying this theorem completes the proof.

□

3. List of Filled-in Information

I followed Evans primarily. I added in many expository details, clarifying remarks, and proof details throughout, but here is a more specific list of most of the ”big” things:

- Motivating the problem with finite dimensional ODE theory and solution properties was my entirely my own.
- I added an interpretation of the semigroup definition to provide a better idea of what it is saying.
- I added an example of semigroups via the heat semigroup.
- I added in multiple extra details in the proof of Proposition 1.1.
- The proof of Theorem 1.2, part 1 is omitted in Evans, so I filled it in (we actually proved that part in functional analysis this semester).
- I added a different proof for showing that $\tilde{R}_\lambda(\lambda - A)u = u$ on $D(A)$ in part 2 of Theorem 1.2.
- I added and proved Proposition 1.3.
- I added in details in the proof of the Hille-Yosida theorem. In particular, I added in the density argument for strong operator convergence, various explanations of well-definedness, extra details on many inequalities, the density argument for extending $S(t)$ from $D(A)$ to all of X, and brief motivation for what we are doing throughout.
- I added in a more general version of the Hille-Yosida theorem, as found in [4].
- I provided a statement of Stone's theorem, a theorem important in quantum mechanics.
- I specifically recast the parabolic problem as the appropriate ODE like in the introduction, and I connected Theorem 2.1 to the actual goal of uniquely solving the parabolic problem.
- In the last theorem, I performed the work in combining the weak formulation and the energy estimate (*) to obtain

$$(\lambda-\gamma)\left\| u\right\|^2_{L^2(U)} \le \left\| f\right\|_{L^2(U)} \left\| u\right\|_{L^2(U)}.$$

I also added in some clarifying remarks, although most of the rest of this theorem was already written with sufficient detail in [1].

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