High Frequency Local Energy Decay for Damped Waves

Collin Kofroth

University of North Carolina at Chapel Hill

April 2, 2022

The Wave and Damped Wave Equations

Define

$$\Box := \frac{\partial^2}{\partial t^2} - \Delta, \qquad \Delta := \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}.$$

The (homogeneous) wave and damped wave equations on $\mathbb{R} \times \mathbb{R}^3$:

$$\begin{cases} \Box u = 0 \\ u(0,x) = f(x) \\ \partial_t u(0,x) = g(x) \end{cases} \qquad \begin{aligned} \Box u + a(x)\partial_t u &= 0 \\ u(0,x) &= f(x) \\ \partial_t u(0,x) &= g(x) \end{aligned}$$

with $a \in C_c^\infty(\mathbb{R}^3)$, $a \geq 0$.

Global Energy

Consider energy functional

$$E[u](t) := \frac{1}{2}\int |\partial u|^2 \, dx, \qquad \partial = (\partial_t, \nabla_x)$$

- u solves wave equation $\implies$ energy conservation
- u solves damped wave equation $\implies$ energy dissipation

Local Energy Decay I

Although energy is conserved, it does decay within compact spatial sets. If u solves wave equation, then

$$\sup_{j\geq 0}\left(\|\langle x\rangle^{-1/2}\partial u\|^2_{L^2(\mathbb{R}_+\times\{\langle x\rangle\approx 2^j\})}+\|\langle x\rangle^{-3/2}u\|^2_{L^2(\mathbb{R}_+\times\{\langle x\rangle\approx 2^j\})}\right)\lesssim E[u](0)$$

provided $n\geq 3$. This is called an integrated local energy estimate.

Utility:

- Scattering problems
- Imply other useful measures of dispersion (Strichartz estimates, pointwise decay estimates)
- Can aid in obtaining long-time existence for nonlinear waves

Local Energy Decay II

Define local energy norms

$$\|u\|_{LE} := \sup_{j\geq 0} \|\langle x\rangle^{-1/2} u\|_{L^2_{t,x}\left(\mathbb{R}_+\times\{\langle x\rangle\approx 2^j\}\right)}$$
$$\|u\|_{LE^1} := \|\partial u\|_{LE} + \|\langle x\rangle^{-1} u\|_{LE}$$
$$\|u\|_{LE^*} := \sum_{j=0}^{\infty} \|\langle x\rangle^{1/2} u\|_{L^2_{t,x}\left(\mathbb{R}_+\times\{\langle x\rangle\approx 2^j\}\right)}.$$

If u solves $\Box u = f$, then

$$\|u\|^2_{LE^1} \lesssim E[u](0) + \|f\|^2_{LE^*}.$$

Full local energy decay (LED) estimate:

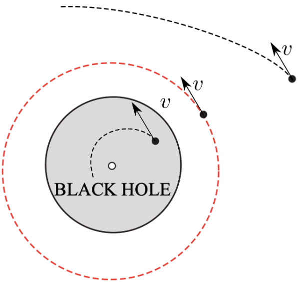
$$\|u\|_{LE^1} + \|\partial u\|_{L^\infty L^2} \lesssim \|\partial u(0)\|_{L^2} + \|f\|_{LE^*+L^1L^2}.$$

Geometric Obstructions

Key obstruction to LED:

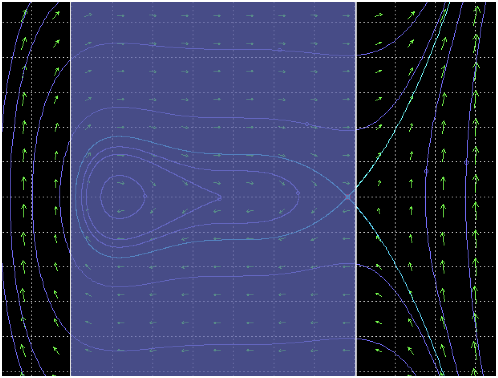
- Trapping: null geodesics stay in a compact set for all time
  - Example: black hole backgrounds
  - Trapping $\implies$ no LED (Ralston 1969, Sbierski 2015)
  - Can recover weaker LED statements for certain types of trapping

Can we recover LED if we use damping to control the trapping?



Geometric Control

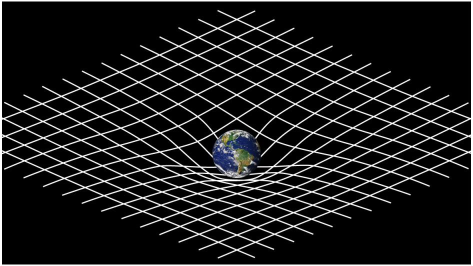
- **Geometric control condition (GCC)**: every bounded null geodesic intersects $\{a > 0\}$ in finite time
- Introduced by Rauch and Taylor (1975) to obtain exponential decay for dissipative equations on compact product manifolds without boundary



Geometric Setting

Modify original problem to

- $(\mathbb{R}^4, g)$ Lorentzian, sgn$(g) = (- + + +)$
- g asymptotically flat
- ∂_t Killing
- Damped wave operator $P = D_\alpha g^{\alpha\beta} D_\beta + iaD_t$, with $a \in C_c^\infty(\mathbb{R}^3_x),\ a \geq 0$



Key Results

- **Previous results** (Bouclet and Royer 2014): LED holds for damped waves satisfying GCC on stationary, asymptotically Euclidean (product) manifolds $\mathbb{R}_t \times \mathbb{R}^n_x$, $n \geq 3$, i.e. metrics

$$g = -dt^2 + g_{ij}(x)dx^i \otimes dx^j.$$

- Example of non-product structure: Kerr space-time
- **New work** (me): allow for full Lorentzian structure

Theorem

Let P be a stationary, AF, damped wave operator satisfying GCC. If ∂_t is uniformly time-like, then LED holds.

Proof Structure

Combine arguments of Metcalfe, Sterbenz, and Tataru for waves on non-trapping Lorentzian space-times (2020) and Bouclet and Royer

- Prove LED in high, medium, and low time-frequency regimes for Schwartz functions
- Combine analysis, perform extension argument

Largely follow MST framework, except where trapping takes place; medium frequencies (Carleman estimates) and low frequencies (elliptic estimates) are unaffected by both trapping and damping

High Frequencies

Trapping occurs at high frequencies, so this is what needs the most modification from MST to avoid loss.

Theorem

Let P be a stationary, AF, damped wave operator satisfying GCC. If ∂_t is uniformly time-like, then the estimate

$$\|u\|_{LE^1} + \|\partial u\|_{L^\infty L^2} \lesssim \|\partial u(0)\|_{L^2} + \|\langle x\rangle^{-2} u\|_{LE} + \|Pu\|_{LE^* + L^1 L^2}$$

holds.

Want to perform positive commutator argument; by microlocal methods, must construct an escape function (and correction term)

Escape Function Construction

Let p, s denote the principal symbols of the self, skew-adjoint parts of P.

Lemma

There exist $q_0, m \in S^0, q_1 \in S^1$, supported in $|\xi| \geq \lambda$, so that

$$H_p q - 2isq + pm \gtrsim \mathbb{1}_{|\xi|\geq\lambda} \langle x \rangle^{-2} (|\xi|^2 + \tau^2),$$

where $q = \tau q_0 + q_1$.

Constructing the symbols

1. On the characteristic set: construct q via factoring argument
   - Interior region: trapping!
     - Semi-bounded geodesics: use a version of GCC (BR)
     - Rest of the interior region
   - Exterior region: perturbation is small, use modified exterior estimates for bootstrapping (MST)
2. On the elliptic set: construct m

Acknowledgments

Thanks to

- Jason Metcalfe (advisor)
- Funding
  - F. Ivy Carroll Summer Research Fellowship
  - NSF grant DMS-2054910 (PI: Jason Metcalfe)
- Conference organizers
- Talk attendees