LOCAL ENERGY DECAY FOR DAMPED WAVES ON STATIONARY, ASYMPTOTICALLY FLAT SPACE-TIMES

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ABSTRACT

Collin Kofroth: Local Energy Decay for Damped Waves on Stationary,
Asymptotically Flat Space-Times
(Under the direction of Jason Metcalfe)

We prove local energy decay for the damped wave equation on stationary, asymptotically flat space-times in $(1 + 3)$-dimensions. Local energy decay constitutes a powerful tool in the study of dispersive partial differential equations on such geometric backgrounds, with applications in areas such as general relativity. By utilizing the geometric control condition to handle trapped trajectories, we are able to recover high frequency estimates without any loss, which we connect to a high energy uniform resolvent estimate. Next, we establish medium and low frequency results, both of which are not affected by the trapping nor the damping. Finally, we combine these analyses together in order to establish local energy decay. This generalizes the integrated version of Bouclet and Royer's work from the setting of asymptotically Euclidean manifolds to the full Lorentzian case.

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TABLE OF CONTENTS

CHAPTER 1: INTRODUCTION **1**

1.1 Background 1

1.2 Problem Setup and Main Results 4

CHAPTER 2: STARTING ENERGY ESTIMATES **14**

CHAPTER 3: HIGH FREQUENCY ANALYSIS **18**

3.1 Introduction 18

3.2 Dynamical Framework 18

3.3 Results on the Flow 28

3.4 Escape Function Construction 36

3.5 Case Reductions 52

3.6 Proof of the High Frequency Estimate 59

3.7 An Application to High Energy Resolvent Estimates 67

CHAPTER 4: MEDIUM FREQUENCY ANALYSIS **72**

4.1 Introduction 72

4.2 Exterior Carleman Estimates 72

4.3 An Interior Carleman Estimate 84

4.4 The Medium Frequency Estimate 95

CHAPTER 5: LOW FREQUENCY ANALYSIS **99**

5.1 Introduction 99

5.2 Weighted Estimates for the Flat Laplacian 100

5.3 Perturbative Estimates 104

5.4 The Low Frequency Estimate 109

CHAPTER 6: ESTABLISHING LOCAL ENERGY DECAY 112

BIBLIOGRAPHY 117

CHAPTER 1

Introduction

1.1 Background

The goal of this work is to establish local energy decay for the damped wave equation on asymptotically flat space-times with stationary metrics subject to the geometric control condition. First, we utilize geometric control to recover the high frequency estimate present in [27] for waves on non-trapping space-times. Since the aforementioned work only utilizes the non-trapping assumption at high frequencies, this establishes the key step in extending time-integrated versions of previously-known results for damped waves on product manifolds (see [6]) to the full Lorentzian setting. From our high frequency estimate, we may apply known results in [27] to conclude local energy decay and complete this extension. We will re-prove the required results of [27] within the framework of damped waves.

Local energy estimates are a collection of rich and well-studied quantities within the field of dispersive partial differential equations, originally introduced on Minkowski space in classical works such as [31, 32, 33], [34]. A particularly important class of local energy estimates are the *integrated local energy estimates*; if u solves the homogeneous flat wave equation

$$(\partial_t^2 - \Delta)u = 0, \qquad \Delta = \sum_{j=1}^{n} \partial_{x_j}^2$$

in spatial dimension $n \geq 3$, then the integrated local energy estimate which we are interested in takes the form

$$\sup_{j\geq 0} \left(\left\| \langle x\rangle^{-1/2} \partial u \right\|_{L_t^2 L_x^2\left(\mathbb{R}_+ \times \{\langle x\rangle \approx 2^j\}\right)} + \left\| \langle x\rangle^{-3/2} u \right\|_{L_t^2 L_x^2\left(\mathbb{R}_+ \times \{\langle x\rangle \approx 2^j\}\right)} \right) \lesssim \|\partial u(0)\|_{L^2}, \tag{1.1}$$

where $\partial = (\partial_t, \nabla)$ denotes the space-time gradient, and $\langle x\rangle = (1 + |x|^2)^{1/2}$ denotes the Japanese bracket of x. This estimate is known to hold in the flat setting through a positive commutator argument using the multiplier introduced in the appendix of [43]. In such a case, we will say that *local energy decay* holds. This is a quantitative statement on *dispersion*, and it heuristically expresses that the energy of the wave must decay quickly enough within compact spatial sets to be integrable in time. Estimates of this form have significant utility, as they have been used to prove other important measures of dispersion such as Strichartz estimates (see [7, 8], [16, 17], [23], [24], [28, 29], [44], [47], and the references therein) and pointwise decay estimates (see [13], [22], [30], [35], [36], [45], and references in these works). Additionally, local energy estimates have applications to nonlinear wave equations where one can develop estimates on an appropriate linearization of the problem, viewing the nonlinearity as a perturbation. These techniques have been applied in many works; see e.g. [4], [18, 19], [25, 26], [42], and the citations contained in them. We will be focused on establishing local energy decay rather than demonstrating its utility via applications.

In [27], the authors proved that local energy decay holds for a broad class of stationary wave operators if and only if

1. *The space-time is non-trapping*: There are no null bicharacteristic rays which stay within a compact set for all time.

2. *The operator satisfies certain spectral assumptions*: Upon replacing time derivatives in the wave operator with a complex parameter, one requires that this family of operators have no eigenvalues in the lower half-plane nor real resonances/embedded eigenvalues (see [27] for more precise definitions); equivalently, one requires analytic continuation of the inverse (*resolvent*) of this family of operators to the entire lower half plane and continuous extension to the real line.

They also established results for *almost* stationary operators, though that is not the context of the work presented here. While the authors employed a non-trapping hypothesis, their work did not require product structure on their space-times, which makes their work highly influential in our own.

Although the absence of trapping is known to be necessary for waves to experience local energy decay (see [39], [41]), one can recover weak local energy decay estimates with a prescribed loss at high frequencies for certain types of trapping (see [9], [10], [14, 15], [24], [37], [47], [48], and the contained references). When the trapping is sufficiently weak/unstable, then this loss is nominal (in fact, logarithmic); this is the case for both the *Schwarzschild* ([24]) and *Kerr* ([47]) space-times. Both space-times possess non-trivial trapped sets, which constitute regions where light remains for all time. Although one can extract weak local energy decay estimates, the trapping still generates an immutable barrier to full local energy decay. We will not be working in a scenario that generates loss, although we would be remiss if we did not briefly mention weak local energy decay and essential space-times that enjoy it.

The study of damped waves also possesses a deep history, especially on compact manifolds. The seminal work [40] introduced the *geometric control condition*, which required that all null bicharacteristic rays intersect the damping region, and they used it to show that the energy of solutions to damped hyperbolic equations on compact product manifolds enjoys exponential decay in time. The uniform exponential bound is equivalent to so-called *strong stabilization*, whereby one can bound the energy at an arbitrary time by the initial energy multiplied by a monotone-decreasing, non-negative function tending to zero as $t \to \infty$. This established the sufficiency of geometric control for strong stabilization in such settings, while [39] demonstrated necessity (also, see [21]). The work [3] showed sufficiency for observability and control on compact manifolds with boundary where the observability/control region is contained within the boundary. While there is notably less literature in the non-compact setting, it was proven in [6] that local energy decay holds for the damped wave equation on asymptotically *Euclidean* space-times with stationary metrics under the assumption of geometric control on trapped geodesics. The authors proved dissipative Mourre estimates to obtain uniform resolvent bounds in different frequency regimes in order to apply a limiting absorption argument. This approach is highly dependent on the stationarity of the metric and the product structure (asymptotically Euclidean metrics contain no metric cross terms).

In this thesis, we combine the approaches of [6] and [27] to establish high frequency local energy estimates for damped waves on stationary, asymptotically flat space-times satisfying the geometric control condition without any loss due to trapping. Then, we will establish the medium and low frequency estimates of [27] and combine the high, medium, and low frequency analyses in order to prove local energy decay. We underscore that we are not requiring the product structure evident in [6] but, instead, allow for the full Lorentzian formulation. Non-product metrics possess non-trivial cross terms and are called *non-static*, of which the Kerr metric constitutes an important example. We most closely keep to the framework present in [27], which does not assume product structure and has results for even more general asymptotically flat non-trapping space-times (such as non-stationary ones). We again stress their use of a non-trapping hypothesis, which we replace by imposing geometric control. Trapping is an intrinsically high-frequency phenomenon, so only their high frequency work is affected by the trapping. Hence, this is the portion of the argument that needs modification to ensure local energy decay, and this is where the influence of [6] comes into play. Since the medium and low frequency analyses (as well as the procedure of combining the different frequency regime estimates into the full local energy decay estimate) do not depend on the non-trapping hypothesis nor use the damping themselves, the corresponding results in [27] readily apply (i.e. our problem essentially becomes a special case here). For the sake of completeness, we will thoroughly perform all of the analysis in the context of damped waves. We also do not need our space-time to be stationary for the medium and low frequency work, so we will prove these without the unnecessary hypothesis.

1.2 Problem Setup and Main Results

Let $(\mathbb{R}^4, g)$ be a Lorentzian manifold with coordinates $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, where g has signature $(- + ++)$. We will consider *damped wave operators* of the form

$$P = \Box_g + iaD_t, \qquad \Box_g = D_\alpha g^{\alpha\beta} D_\beta,$$

where $a \in C_c^\infty(\mathbb{R}^3)$ is non-negative and positive on an open set, and $D_\alpha = \frac{1}{i}\partial_\alpha,\ \alpha = 0, 1, 2, 3$. Greek indices will generally range over such values, whereas Latin indices will run over the integers $1, 2$, and 3. Notice that we are using the standard Einstein summation convention, which we will do throughout this work. We will also subject g to an *asymptotic flatness* condition. More precisely, we first define the norm

$$\|h\|_{AF} = \sum_{|\alpha|\le 2} \left\| \langle x\rangle^{|\alpha|}\, \partial^\alpha h \right\|_{\ell^1_j L^\infty([0,T]\times A_j)},$$

where $A_j = \{\langle x\rangle \approx 2^j\}$ for $j \ge 0$ denote inhomogeneous dyadic regions, and ℓ^1_j denotes the ℓ^1 norm over the j index. The notation $A \lesssim B$ means that $A \le CB$ for some $C > 0$, and the notation $A \approx B$ means that $B \lesssim A \lesssim B$. In the definition of the A_j's, we require that these implicit constants are compatible to cover $\mathbb{R}^3$. That is,

$$\bigcup_{j=0}^{\infty} A_j = \mathbb{R}^3.$$

This allows us to define the AF topology.

Definition 1.1. We say that P is *asymptotically flat* if $\|g-m\|_{AF} < \infty$, where m denotes the Minkowski metric, and

$$\left\| \langle x \rangle^{|\alpha|} \partial^\alpha g \right\|_{\ell^1_j L^\infty([0,T]\times A_j)} \lesssim_\alpha 1$$

for all $\alpha \in \mathbb{N}^3$ with $|\alpha| \geq 3$.

The latter condition will be necessary for certain functions appearing in this work to be symbolic in the Kohn-Nirenberg sense (we will define this in Section 3.2). We remark that the dyadic summability assumptions on our metric are weaker than the long-range perturbation condition present in [6] (which provides a symbolic-type decay estimate for derivatives of the metric in x in terms of $\langle x \rangle^{-\rho}$, with $\rho > 0$ fixed). We will primarily be interested in when g is *stationary* with Killing field ∂_t (in this paper, stationary metrics will be assumed to have this Killing field). In this case, we will call the operator P a *stationary, asymptotically flat damped wave operator.*

Next, we introduce

- the parameters R_0 and $\mathbf{c}$, which are such that

$$\|g-m\|_{AF_{>R_0}} \leq \mathbf{c} \ll 1,$$

  where the subscript denotes the restriction of the norm to $\{|x| > R_0\}$. The parameter $\mathbf{c}$ should be viewed as being fixed first, after which we find an R_0 for which the above holds. Without loss of generality, we will assume that $\operatorname{supp} a \subset \{|x| \leq R_0\}$ (as it is unnecessarily beneficial outside of this set).

- the sequence $(c_j)_{j \geq \log_2 R_0}$ satisfying

$$\|g-m\|_{AF(A_j)} \lesssim c_j, \qquad \sum_j c_j \lesssim \mathbf{c},$$

where $\|\cdot\|_{AF(A_j)}$ denotes the restriction of the AF norm to the dyadic region A_j. We may assume without any loss of generality that the sequence is slowly-varying, i.e.

$$c_j/c_k \leq 2^{\delta|k-j|}, \qquad \delta \ll 1.$$

These parameters tell us that, outside of a large enough spatial ball, the operator P is a uniformly small perturbation of the flat wave operator $\Box_m = \partial_t^2 - \Delta$ (which we simply denote as $\Box$). The sequence (c_j) provides a quantitative measure on the size of the AF norm throughout each spatial dyadic region outside of this ball.

We will also assume throughout that the vector field ∂_t is uniformly time-like, which essentially constitutes a choice of coordinates. This condition, coupled with the signature of the metric, ensures that $D_i g^{ij} D_j$ is uniformly elliptic, i.e.

$$g^{ij}\xi_i\xi_j \approx |\xi|^2, \qquad \xi \neq 0, \tag{1.2}$$

where $|\cdot|$ denotes the standard Euclidean norm. This follows from the positive-definiteness of the momentum-energy tensor

$$Q[\varphi] = d\varphi \otimes d\varphi - \frac{1}{2}g^{-1}(d\varphi, d\varphi)g$$

associated to smooth functions $\varphi : \mathbb{R}^4 \to \mathbb{R}$ when applied to time-like vector fields (see [2]). Indeed, if $\varphi(x) = \xi_j x^j$, then one readily computes that

$$Q[\xi_j x^j] = (\xi_i dx^i) \otimes (\xi_j dx^j) - \frac{1}{2}g^{ij}\xi_i\xi_j g.$$

Since ∂_t is uniformly time-like, $g_{00} \lesssim -1$. Further, $Q[\xi_j x^j]$ is positive-definite in ∂_t, and so

$$-\frac{1}{2}g_{00}g^{ij}\xi_i\xi_j = Q[\xi_j x^j](\partial_t, \partial_t) \gtrsim |\xi|^2.$$

In particular, we get the desired lower bound for ellipticity. The Cauchy-Schwarz inequality and the boundedness of the metric provide the upper bound.

Next, we define the local energy norms

$$\|u\|_{LE} = \sup_{j\geq 0} \left\| \langle x\rangle^{-1/2} u \right\|_{L_t^2 L_x^2 \left(\mathbb{R}_+ \times A_j\right)},$$
$$\|u\|_{LE^1} = \|\partial u\|_{LE} + \left\|\langle x\rangle^{-1} u\right\|_{LE}.$$

A predual-type norm to the LE norm is the LE^* norm, which is defined as

$$\|f\|_{LE^*} = \sum_{j=0}^{\infty} \left\| \langle x\rangle^{1/2} f \right\|_{L_t^2 L_x^2 \left(\mathbb{R}_+ \times A_j\right)}.$$

Here, $L_t^p L_x^q$ denotes the Bochner space $L^p(\mathbb{R}_+, L^q(\mathbb{R}^3))$. In the particular case of $p, q = 2$, then this is a Hilbert space; we will use $\langle \cdot, \cdot \rangle$ to denote its inner product. Lastly, we define the sum-space norm

$$\|f\|_{LE^* + L_t^1 L_x^2} = \inf_{f = f_1 + f_2} \left(\|f_1\|_{LE^*} + \|f_2\|_{L_t^1 L_x^2} \right).$$

If we wish for the time interval to be e.g. $[0, T]$ in the above norms, then we will use the notation $\|u\|_{LE[0,T]}$, $\|u\|_{LE^1[0,T]}$, $\|u\|_{LE^*[0,T]}$, $\|u\|_{LE^*+L_t^1 L_x^2[0,T]}$ (although we will write $LE^*[0,T] + L_t^1 L_x^2[0,T]$ when referring to this space outside of norm subscripts), etc. Subcripting any of these spaces with a zero (e.g. LE_0^1) denotes the closure of C_c^∞ in the relevant space. A subscript of c on any of these spaces denotes compact spatial support.

There are two additional function spaces that will be utilized extensively in this work. The first is the class of Schwartz functions $\mathcal{S}(\mathbb{R}^4)$, which will be useful for approximation arguments. The second is a particular collection of functions which is often the natural class to study wave equations.

Definition 1.2. Let $T > 0$. We define the class $\mathcal{W}_T$ to be the space of all functions $u \in C^2([0,T] \times \mathbb{R}^3)$ for which there exists $R > 0$ so that $u(t,x) = 0$ for all $t \in [0,T]$ and $|x| > R$. That is,

$$\mathcal{W}_T = \{u \in C^2([0,T] \times \mathbb{R}^3) : (\exists R > 0)(\forall |x| > R)(\forall t \in [0,T]) \;\; u(t,x) = 0\}.$$

We are interested in Cauchy problems of the form

$$\begin{cases} Pu = f \in LE^*[0,T] + L_t^1 L_x^2[0,T], \\ u[0] = (u(0), \partial_t u(0)) \in \dot{H}^1 \oplus L^2. \end{cases}$$

Remark 1.3. The decay conditions on $u \in \mathcal{W}_T$ are not as restrictive as they might initially appear. If the Cauchy data is compactly-supported, then the condition is free by finite speed of propagation. If it is not, then one can approximate the data (which generically lives in the energy space) by compactly-supported data. The regularity conditions on u are also not restrictive, as one can perform density arguments to reduce to the case of increased regularity. ■

Now, we state the pertinent local energy estimates for such problems.

Definition 1.4. We say that *local energy decay* holds for an asymptotically flat wave operator if the following estimate holds for all $T > 0$:

$$\|u\|_{LE^1[0,T]} + \|\partial u\|_{L_t^\infty L_x^2[0,T]} \lesssim \|\partial u(0)\|_{L^2} + \|Pu\|_{LE^* + L_t^1 L_x^2[0,T]} \tag{1.3}$$

for all $u \in \mathcal{W}_T$ such that $u[0] \in \dot{H}^1 \oplus L^2$, with the implicit constant being independent of T.

The notion of an asymptotically flat wave operator is more broad than an asymptotically flat damped wave operator. They need not feature a damping term, and they are allowed to possess general lower-order terms which are asymptotically flat in an appropriate sense (see Definition 1.1 in [27] for a precise definition).

Note that, due to global energy conservation for the flat wave problem, the general definition of local energy decay that we have given here is consistent with the integrated local energy estimate for the flat wave equation in (1.1) (in the inhomogeneous case, one applies Hölder's inequality to the forcing). This estimate is known to hold whenever P is a small asymptotically flat perturbation of $\Box$ (see [1], [25, 26], [29]). In [27], the authors considered large AF perturbations and proved that, for stationary metrics, the local energy decay estimate (1.3) is equivalent to assuming that the wave operator P is non-trapping and has no negative eigenfunctions (L^2 eigenfuctions with corresponding eigenvalues in the lower half-plane) nor real resonant states (outgoing non-L^2 eigenfunctions with real "eigenvalues," which are called resonances); see Definitions 2.2, 2.4, and 2.8 in [27] for more precise definitions of these objects. The non-trapping hypothesis only arose during their proof of a high frequency estimate (Theorem 2.11 in [27]), which took the form

$$\|u\|_{LE^1[0,T]} + \|\partial u\|_{L^\infty_t L^2_x[0,T]} \lesssim \|\partial u(0)\|_{L^2} + \left\| \langle x \rangle^{-2} u \right\|_{LE[0,T]} + \|Pu\|_{LE^* + L^1_t L^2_x[0,T]} . \tag{1.4}$$

The implicit constant in the above estimate is crucially independent of T. This estimate does not require u to be truncated to large time frequencies, but this is the context in which it was used in proving local energy decay.

The added spatial weight in the error term does not play a particular role in making this high frequency. Rather, it is the weight that naturally arises when performing a bootstrapping argument in the proof of the estimate; it is largely unimportant since this estimate can be reduced to studying solutions with compact spatial support.

Remark 1.5. To see this as an estimate on the high frequencies, let $u \in \mathcal{S}(\mathbb{R}^4)$ be frequency-supported in time for τ in the range $1 \ll \tau_1 \leq |\tau| < \infty$ (we will only apply the estimate for such u in our proof of local energy decay). Then, we can use Plancherel's theorem in t to obtain that

$$\begin{aligned} \left\| \langle x \rangle^{-2} u \right\|_{LE_{t,x}} &\approx \left\| \langle x \rangle^{-2} \hat{u}(\tau, x) \right\|_{LE_{\tau,x}} \lesssim \frac{1}{\tau_1} \left\| \langle x \rangle^{-2} \tau \hat{u}(\tau, x) \right\|_{LE_{\tau,x}} \\ &\approx \frac{1}{\tau_1} \left\| \langle x \rangle^{-2} \partial_t u \right\|_{LE_{t,x}} \lesssim \frac{1}{\tau_1} \|u\|_{LE^1_{t,x}} . \end{aligned} \tag{1.5}$$

For large enough τ_1, this term can be absorbed into the left-hand side of (1.4), providing local energy decay for solutions restricted to high frequencies.

In fact, we may apply the high frequency estimate (1.4) to $u(t - T/2)$ to get (after dropping the uniform energy piece) that

$$\|u\|_{LE^1[-T/2,T/2]} \lesssim \|\partial u(-T/2)\|_{L^2} + \left\| \langle x \rangle^{-2} u \right\|_{LE[-T/2,T/2]} + \|Pu\|_{LE^*[-T/2,T/2]} .$$

Since the implicit constant is independent of T, we may take the limit as $T \to \infty$ and apply the prior work in (1.5) to obtain that

$$\|u\|_{LE^1} \lesssim \|Pu\|_{LE^*} .$$

This is the context that we will apply the estimate to establish local energy decay. ■

Our first main theorem is the following, which states that we recover the high frequency estimate (1.4) of [27] when working with damped waves and replacing the non-trapping hypothesis with the geometric control condition.

Theorem 1.6. *Let P be a stationary, asymptotically flat damped wave operator satisfying the geometric control condition, and suppose that ∂_t is uniformly time-like. Then, the high frequency local energy estimate (1.4) holds for all for all $u \in \mathcal{W}_T$ such that $u[0] \in \dot{H}^1 \oplus L^2$. The implicit constant is independent of T.*

The geometric control condition, initially introduced in [40] for dissipative hyperbolic equations on compact product manifolds, requires that every trapped null bicharacteristic ray intersects the damping region. We will make this more precise in Section 3.2.

Remark 1.7. The implicit constant in the bound depends on R_0. In fact, much of our work will implicitly depend on R_0 due to our applications of asymptotic flatness. It is essential to note that this parameter is fixed second (with $\mathbf{c}$ being fixed first), after which our other parameters (such as the scaling parameter γ and the high-frequency parameter λ which will both be introduced in Chapter 3) will be chosen (and hence depend on it). We will not track the dependence on R_0 within our implicit constants any longer. Our constants throughout will *not* depend on T, however. ■

Our second main theorem is local energy decay.

Theorem 1.8. *Let P be a stationary, asymptotically flat damped wave operator satisfying the geometric control condition, and suppose that ∂_t is uniformly time-like. Then, local energy decay holds, with the implicit constant in (1.3) independent of T.*

This follows rather directly from our high frequency estimate and the existing work in [27], but we will reproduce the necessary analysis, with added details, within our setting.

The structure of this thesis is as follows:

- **Chapter 2.** This section contains various energy inequalities which we will use in many subsequent sections. In particular, we will prove a uniform energy estimate, and we will cite two exterior estimates from [27]. We will not prove the exterior estimates here since the damping is zero in the exterior, and so the proofs in [27] do not require any modification.

- **Chapter 3.** This section will contain our high frequency work. In Section 3.2, we will introduce the Hamiltonian formalism required to define trapping and geometric control, then we will state a key lemma (Lemma 3.4) for the proof of Theorem 1.6; namely, we construct an appropriate escape function and lower-order correction to allow for a positive commutator argument proof of the theorem. In Section 3.3, we demonstrate various results on the bicharacteristic flow that are vital for proving Lemma 3.4, which we prove in Section 3.4. In Section 3.5, we will establish multiple case reductions to simplify the proof of Theorem 1.6, which we prove in Section 3.6. Then, we will provide an application to a uniform high energy resolvent bound in Section 3.7. This chapter is the most significant portion of our work, in terms of ingenuity.

- **Chapter 4.** This section will contain our medium frequency work. This work is based on *Carleman estimates*, which are spatially-weighted $L_t^2 L_x^2$ norms with a pseudoconvex weight function. This will yield local energy decay for solutions supported at time frequencies in any range bounded away from both zero and infinity. Our main estimate will require two auxiliary Carleman estimates, one which applies within a compact set (see Section 4.3) and one which applies on the exterior of a compact set (see Section 4.2). None of the work here requires the metric to be non-trapping, nor does the damping play a role. Additionally, one does not need the metric to be stationary.

- **Chapter 5.** This section will contain our low frequency work. We will analyze our operator at time frequency zero and obtain results based on weighted estimates of the flat Laplacian Δ. Such work will provide us with local energy decay for our damped wave operator in a small neighborhood of time frequency zero. As with Chapter 4, the trapping, damping, and stationarity of the metric play no role here.

- **Chapter 6.** This section contains our proof of Theorem 1.8. First, we prove a simplified result using our high, medium and low frequency analyses where we remove the Cauchy data at times 0 and T, then we show that this implies Theorem 1.8.

For the remainder of paper, we will fix the cutoffs

$$\chi \in C_c^\infty \text{ non-increasing, } \chi \equiv 1 \text{ for } |x| \leq 1, \ \chi \equiv 0 \text{ for } |x| > 2,$$

$$\chi_{<R}(|x|) = \chi\left(\frac{|x|}{R}\right), \qquad \chi_{>R} = 1 - \chi_{<R},$$

and

$$\chi_R \in C_c^\infty, \ 0 \leq \chi_R \leq 1, \ \operatorname{supp} \chi_R \subset \{|x| \approx R\}.$$

We will occasionally write $r = |x|$. When working in frequency variables, we will often add the variable into the subscript to make the dependence clear (e.g. $\chi_{|\xi|>\lambda}$).

CHAPTER 2

Starting Energy Estimates

In this section, we will establish or cite various energy estimates which will be useful throughout this work. Our starting point is a standard uniform energy inequality.

Proposition 2.1. *Let P be a stationary damped wave operator, ∂_t be uniformly time-like, and $T > 0$. Then, we have the estimate*

$$\|\partial u(t)\|_{L^2}^2 \lesssim \|\partial u(0)\|_{L^2}^2 + \int_0^T \int_{\mathbb{R}^3} |Pu\ \partial_t u|\, dxdt, \qquad 0 \le t \le T \tag{2.1}$$

for all $u \in \mathcal{W}_T$.

Proof. Call $Pu = f$, and define the energy functional

$$E[u](t) = \int_{\mathbb{R}^3} D_i g^{ij} D_j u\bar{u} - g^{00}|\partial_t u|^2\, dx.$$

After integrating the first term by parts, this functional is readily seen to be coercive due to the uniformly time-like nature of ∂_t, i.e.

$$E[u](t) \approx \|\partial u(t)\|_{L^2}^2 .$$

Differentiating in t and integrating by parts,

$$\begin{aligned}\frac{d}{dt}E[u](t) &= -\int_{\mathbb{R}^3} g^{00}(\partial_t^2 u\partial_t\bar{u} + \partial_t u\partial_t^2\bar{u})\, dx + \int_{\mathbb{R}^3} D_i g^{ij} D_j \partial_t u\bar{u} + D_i g^{ij} D_j u\partial_t\bar{u}\, dx \\ &= \int_{\mathbb{R}^3} (g^{00}D_t^2 + D_i g^{ij} D_j)u\partial_t\bar{u} + \partial_t u\overline{(g^{00}D_t^2 + D_i g^{ij} D_j)u}\, dx\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} \left(-(g^{0j} D_j D_t + D_j g^{0j} D_t + iaD_t)u + f\right) \partial_t \bar{u} \\
&\qquad + \partial_t u \overline{(-(g^{0j} D_j D_t + D_j g^{0j} D_t + iaD_t)u + f)}\, dx \\
&= 2\mathrm{Re} \int_{\mathbb{R}^3} \bar{f} \partial_t u \, dx - 2 \int_{\mathbb{R}^3} a |\partial_t u|^2 \, dx.
\end{aligned}
$$

Dropping the damping term and integrating the resulting estimate in time yields the inequality

$$
E[u](t) \lesssim E[u](0) + \int_0^T \int_{\mathbb{R}^3} |f \partial_t u| \, dx dt, \qquad 0 \le t \le T.
$$

Applying the coercivity allows us to conclude. □

Remark 2.2. Note that when $Pu = 0$, we have the energy dissipation statement

$$
\frac{d}{dt} E[u](t) = -2 \int_{\mathbb{R}^3} a |\partial_t u|^2 \, dx \le 0.
$$

■

Applying the Schwarz inequality to Proposition 2.1 provides us with estimates which will prove useful throughout this work.

Corollary 2.3. *Under the same assumptions as Proposition 2.1, the uniform energy estimates*

$$
\|\partial u\|_{L^\infty_t L^2_x} \lesssim \|\partial u(0)\|_{L^2} + \|Pu\|_{L^1_t L^2_x},
$$

$$
\|\partial u\|_{L^\infty_t L^2_x} \lesssim \|\partial u(0)\|_{L^2} + \|Pu\|_{LE^*}^{1/2} \|u\|_{LE^1}^{1/2},
$$

and

$$
\|\partial u\|_{L^\infty_t L^2_x} \lesssim \|\partial u(0)\|_{L^2} + \varepsilon^{-1} \|Pu\|_{LE^* + L^1_t L^2_x} + \varepsilon \|u\|_{LE^1}, \qquad \forall \varepsilon > 0
$$

hold.

Proof. By Proposition 2.1, we have the estimate

$$\|\partial u(t)\|_{L^2}^2 \lesssim \|\partial u(0)\|_{L^2}^2 + \int_0^T \int_{\mathbb{R}^3} |Pu\partial_t u| \, dxdt, \qquad 0 \le t \le T.$$

To obtain the first estimate that we claimed, one applies the Schwarz inequality, takes a supremum in time of $\partial_t u$, and uses Young's inequality for products. To obtain the second estimate, write

$$|Pu \ \partial_t u| = \left(\langle x\rangle^{1/2} |Pu|\right) \left(\langle x\rangle^{-1/2} |\partial_t u|\right)$$

in (2.1) and apply the Schwarz inequality and Hölder's inequality applied to ℓ^1 with conjugate exponents $(p,q) = (1,\infty)$. The third estimate follows from splitting $Pu = f_1 + f_2$ with $f_1 \in L_t^1 L_x^2[0,T]$ and $f_2 \in LE^*[0,T]$, applying the work for the first estimate to f_1 and the work for the second estimate to f_2, then using Young's inequality with parameter $\varepsilon > 0$. □

Next, we cite two useful exterior estimates from [27]. Both of these are relevant when we are in the region of space where our operator P is a small AF perturbation of $\Box$, in which case we obtain good energy estimates with a necessary truncation error (each will feature different errors). Since the damping is identically zero in this region, these results hold without any proof modifications, so we omit them here.

The first estimate will be used in the low frequency regime.

Proposition 2.4 (Proposition 3.1 in [27])**.** *Suppose that P is asymptotically flat and $R \ge R_0$. Then,*

$$\|u\|_{LE^1_{>R}} \lesssim \|\partial u(0)\|_{L^2_{>R}} + \|\partial u\|_{LE_R} + \|Pu\|_{LE^*_{>R}}. \tag{2.2}$$

The proof involves estimating u with an extension of its average inside of the R-annulus. Notice that we have an error in the form of a weighted L^2 norm of ∂u. The next estimate removes the derivative from this error at the expense of the energy at time T. It will be used in the high and medium frequency regimes. This estimate is also similar to work in [23] on the Schrödinger equation.

Proposition 2.5 (Proposition 3.2 in [27]). *If P is asymptotically flat and $R \geq R_0$, then*

$$\|u\|_{LE^1_{>R}} \lesssim \|\partial u(0)\|_{L^2_{>R}} + \|\partial u(T)\|_{L^2_{>R}} + R^{-1}\|u\|_{LE_R} + \|Pu\|_{LE^*_{>R}}. \tag{2.3}$$

Their proof is a positive commutator argument using the multiplier $Q_1 + Q_2$, where

$$Q_1 = \chi_{>2R}(|x|)f(|x|)\frac{x_j}{|x|}g^{jk}D_k + D_k\chi_{>2R}(|x|)f(|x|)\frac{x_j}{|x|}g^{jk}$$

is the principal term, and

$$Q_2 = \chi_{>2R}(|x|)f'(|x|)$$

is the lower-order correction term. Here, $f(|x|) = \dfrac{|x|}{|x| + 2^j}$, and j is chosen so that $2^j \geq R$. Detailed proofs of both of these results can be found in e.g. [27], [5].

CHAPTER 3

High Frequency Analysis

3.1 Introduction

In this section, we will establish Theorem 1.6. The notions of trapping and geometric control are intrinsically dynamical, so we will provide a thorough discussion of the relevant theory. Namely, we must introduce the bicharacteristic flow generated by the principal symbol of the damped wave operator and the properties that it satisfies. From here, we will construct an escape function and correction term in order to utilize a positive commutator argument and prove the theorem. The constructed symbols will satisfy an appropriate positivity bound, which will allow us to apply the sharp Gårding inequality upon swapping to the framework of pseudodifferential operators. The symbols will also be supported in an unbounded range of frequencies $[\lambda, \infty)$ with $\lambda \gg 1$. This will be fundamentally important for bootstrapping error terms resulting from employment of pseudodifferential calculus.

3.2 Dynamical Framework

In order to state the geometric control condition more precisely, we must first outline our dynamical framework, which is rooted in the Hamiltonian dynamics pertaining to the principal symbol of the operator P. Since we assumed that ∂_t is uniformly time-like, the signature of the metric and the cofactor expansion for the inverse metric tells us that $g^{00} \lesssim -1$, as well. This allows us to divide through by g^{00} and preserve the assumptions on the operator coefficients (see [29]). Hence, we may assume (without loss of generality) that $g^{00} = 1$. Note that since g^{00} was initially negative, dividing by it swaps the signs of all non-zero metric terms. For this reason, we will re-label $g^{0j}(g^{00})^{-1}$ and $g^{ij}(g^{00})^{-1}$ as $-g^{0j}$ and $-g^{ij}$, respectively. After these modifications, the principal symbol of P is

$$p(\tau, x, \xi) = \tau^2 - 2\tau g^{0j}(x)\xi_j - g^{ij}(x)\xi_i\xi_j.$$

This is considered as a smooth function on $T^*\mathbb{R}^4 \setminus o$, where o denotes the zero section. This symbol generates a bicharacteristic/Hamiltonian flow on $\mathbb{R} \times T^*\mathbb{R}^4$ given by

$$\varphi_s(w) = (t_s(w), \tau_s(w), x_s(w), \xi_s(w))$$

which solves

$$\begin{cases} \dot{t}_s = \partial_\tau p(\varphi_s(w)), \\ \dot{\tau}_s = -\partial_t p(\varphi_s(w)), \\ \dot{x}_s = \nabla_\xi p(\varphi_s(w)), \\ \dot{\xi}_s = -\nabla_x p(\varphi_s(w)) \end{cases}$$

with initial data $w \in T^*\mathbb{R}^4$. Explicitly, one can write the system as

$$\begin{cases} \dot{t}_s = 2\tau_s - 2g^{0j}(x_s)[\xi_s]_j, \\ \dot{\tau}_s = 0, \\ (\dot{x}_s)_k = -2\tau_s g^{0k}(x_s) - 2g^{kj}(x_s)[\xi_s]_j, \\ (\dot{\xi}_s)_k = 2\tau_s \partial_{x_k} g^{0j}(x_s)[\xi_s]_j + \partial_{x_k} g^{ij}(x_s)[\xi_s]_i[\xi_s]_j. \end{cases}$$

Since g is smooth and asymptotically flat, and ∂_t is uniformly time-like, we have a unique, smooth, globally-defined flow with smooth dependence on the data. We will have particular interest in *null* bicharacteristics, i.e. those with initial data lying in the zero set of p (also called the *characteristic set* of P and denoted $\mathrm{Char}(P)$). Using the flow φ_s, we define the *forward* and *backward trapped* and *non-trapped* sets with respect to φ_s, respectively, as

$$\Gamma_{tr} = \left\{ w \in T^*\mathbb{R}^4 \setminus o : \sup_{s \geq 0} |x_s(w)| < \infty \right\} \cap \mathrm{Char}(P),$$
$$\Lambda_{tr} = \left\{ w \in T^*\mathbb{R}^4 \setminus o : \sup_{s \geq 0} |x_{-s}(w)| < \infty \right\} \cap \mathrm{Char}(P),$$
$$\Gamma_\infty = \left\{ w \in T^*\mathbb{R}^4 \setminus o : |x_s(w)| \to \infty \text{ as } s \to \infty \right\} \cap \mathrm{Char}(P),$$
$$\Lambda_\infty = \left\{ w \in T^*\mathbb{R}^4 \setminus o : |x_{-s}(w)| \to \infty \text{ as } s \to \infty \right\} \cap \mathrm{Char}(P).$$

The *trapped* and *non-trapped* sets are defined as

$$\Omega_{tr}^{p} = \Gamma_{tr} \cap \Lambda_{tr},$$
$$\Omega_{\infty}^{p} = \Gamma_{\infty} \cap \Lambda_{\infty},$$

respectively.

Definition 3.1. The flow is said to be *non-trapping* if $\Omega_{tr}^{p} = \emptyset$.

Now, we may state the geometric control condition precisely. Recall that our damping function was denoted a.

Definition 3.2. We say that geometric control holds if

$$(\forall w \in \Omega_{tr}^{p})(\exists s \in \mathbb{R}) \;\; a(x_s(w)) > 0. \tag{3.1}$$

In contrast to the definition in [40] (given in Assumption (A)), we apply this condition specifically to the trapped null bicharacteristics (since *all* null bicharacteristics are trapped when the manifold is compact, such a specification was unnecessary in [40]). We will assume that (3.1) holds. Note that if (3.1) holds and $a \equiv 0$, then Ω_{tr}^{p} must be empty, meaning that the flow is non-trapping. In this case, we are back in the setting of [27]. For this reason, we will assume that $a > 0$ on an open set.

It will be beneficial to utilize a scaling property of P. Given a solution u to $Pu = f$, consider

$$\tilde{v}(t,x) := \gamma^{-2} u(\gamma t, \gamma x), \qquad \gamma > 0.$$

If we call

$$\tilde{P} = D_\alpha \tilde{g}^{\alpha\beta} D_\beta + i\gamma \tilde{a} D_t, \qquad \tilde{g}^{\alpha\beta}(x) = g^{\alpha\beta}(\gamma x), \qquad \tilde{a}(x) = a(\gamma x),$$

then $\tilde{v}$ solves

$$\tilde{P}\tilde{v} = \tilde{f}, \qquad \tilde{f}(t,x) = f(\gamma t, \gamma x)$$

if and only if u solves $Pu = f$ (we can similarly undo the scaling to move between the frameworks). Notice that the scaled problem allows for an arbitrarily large constant γ in front of the damping.

Analogous Hamiltonian systems and trapped sets exist for the principal symbol $\tilde{p}$ of $\tilde{P}$, and this amounts to simply replacing g by $\tilde{g}$. If we assume that geometric control holds for the flow generated by p, then we must check that it holds for the scaled problem.

Proposition 3.3. *Assume that (3.1) holds. Then, for any $\gamma > 0$, (3.1) holds for the flow generated by $\tilde{p}$, with a replaced by $\tilde{a}$.*

Note that since $g^{00} \equiv 1$, it follows that $\tilde{g}^{00} \equiv 1$.

Proof. The flow generated by $\tilde{p}$ solves the system

$$\begin{cases} \dfrac{d}{ds}\tilde{t}_s = 2\tilde{\tau}_s - 2\tilde{g}^{0j}(\tilde{x}_s)[\tilde{\xi}_s]_j, \\ \dfrac{d}{ds}\tilde{\tau}_s = 0, \\ \dfrac{d}{ds}(\tilde{x}_s)_k = -2\tilde{\tau}_s\tilde{g}^{0k}(\tilde{x}_s) - 2\tilde{g}^{kj}(\tilde{x}_s)[\tilde{\xi}_s]_j, \\ \dfrac{d}{ds}(\tilde{\xi}_s)_k = 2\tilde{\tau}_s\partial_{x_k}\tilde{g}^{0j}(\tilde{x}_s)[\tilde{\xi}_s]_j + \partial_{x_k}\tilde{g}^{ij}(\tilde{x}_s)[\tilde{\xi}_s]_i[\tilde{\xi}_s]_j, \\ (\tilde{t}_s, \tilde{\tau}_s, \tilde{x}_s, \tilde{\xi}_s)\big|_{s=0} = (t, \tau, x, \xi). \end{cases}$$

Applying the chain rule and multiplying through by γ provides us with the system

$$\begin{cases} \dfrac{d}{ds}(\gamma\tilde{t}_s) = 2(\gamma\tilde{\tau}_s) - 2g^{0j}(\gamma\tilde{x}_s)[\gamma\tilde{\xi}_s]_j, \\ \dfrac{d}{ds}(\gamma\tilde{\tau}_s) = 0, \\ \dfrac{d}{ds}(\gamma\tilde{x}_s)_k = -2(\gamma\tilde{\tau}_s)g^{0k}(\gamma\tilde{x}_s) - 2g^{kj}(\gamma\tilde{x}_s)[\gamma\tilde{\xi}_s]_j, \\ \dfrac{d}{ds}(\gamma\tilde{\xi}_s)_k = 2(\gamma\tilde{\tau}_s)[(\partial_{x_k}g^{0j})(\gamma\tilde{x}_s)][\gamma\tilde{\xi}_s]_j + [(\partial_{x_k}g^{ij})(\gamma\tilde{x}_s)][\gamma\tilde{\xi}_s]_i[\gamma\tilde{\xi}_s]_j, \\ \big((\gamma\tilde{t})_s, (\gamma\tilde{\tau})_s, (\gamma\tilde{x})_s, (\gamma\tilde{\xi})_s\big)\big|_{s=0} = (\gamma t, \gamma\tau, \gamma x, \gamma\xi). \end{cases}$$

This is the same system that is solved by the Hamiltonian flow generated by p with initial data $(\gamma t, \gamma\tau, \gamma x, \gamma\xi)$.

By uniqueness, we can conclude that

$$\gamma\tilde{t}_s(t,\tau,x,\xi) = t_s(\gamma t,\gamma\tau,\gamma x,\gamma\xi)$$

$$\gamma\tilde{\tau}_s(t,\tau,x,\xi) = \tau_s(\gamma t,\gamma\tau,\gamma x,\gamma\xi)$$

$$\gamma\tilde{x}_s(t,\tau,x,\xi) = x_s(\gamma t,\gamma\tau,\gamma x,\gamma\xi)$$

$$\gamma\tilde{\xi}_s(t,\tau,x,\xi) = \xi_s(\gamma t,\gamma\tau,\gamma x,\gamma\xi).$$

Now, let $w = \Omega_{tr}^{\tilde{p}}$. From the above, we have that

$$\tilde{x}_s(w) = \gamma^{-1}x_s(\tilde{w}), \qquad \tilde{w} = \gamma w.$$

Since

$$\sup_{s\in\mathbb{R}} |x_s(\tilde{w})| = \gamma \sup_{s\in\mathbb{R}} |\tilde{x}_s(w)| < \infty,$$

it follows that $\tilde{w} \in \Omega_{tr}^{p}$. By (3.1), there exists $s' \in \mathbb{R}$ so that $a(x_{s'}(\tilde{w})) > 0$, and so

$$\tilde{a}(\tilde{x}_{s'}(w)) = a(\gamma\tilde{x}_{s'}(w)) = a(x_{s'}(\tilde{w})) > 0,$$

which completes the proof. □

Now that we have shown that geometric control is invariant under scaling, we will fix a large $\gamma > 0$ and study the problem from the scaled perspective (where our damping is now multiplied by γ) while reverting back to our original notation (x and ξ, no tildes, etc.). More precise conditions on the size of γ will come in Section 3.4. It is readily seen that it is equivalent to prove Theorem 1.6 for the scaled problem, where we now have a large constant in front of the damping term.

Our proof of Theorem 1.6 is a positive commutator argument. At the symbolic level, this requires the construction of an escape function (as well as a lower-order correction). We must consider the skew-adjoint contribution of P, which will be a purely beneficial term due to the presence of the damping. Let p and s_{skew} represent the principal symbols of the self and skew-adjoint parts of P, respectively. Namely,

$$p(\tau,x,\xi) = \tau^2 - 2\tau g^{0j}(x)\xi_j - g^{ij}(x)\xi_i\xi_j$$
$$s_{skew}(\tau,x,\xi) = i\gamma\tau a(x).$$

The multiplication by γ in s_{skew} will prove advantageous for a bootstrapping argument, which is precisely why we implement the γ-scaling. Now, we are ready to state our escape function result, which we will prove in Section 3.4.

Lemma 3.4. *For all $\lambda > 1$, there exist symbols $q_j \in S^j(T^*\mathbb{R}^3)$ and $m \in S^0(T^*\mathbb{R}^3)$, all supported in $|\xi| \geq \lambda$, so that*

$$H_p q - 2is_{skew}q + pm \gtrsim \mathbb{1}_{|\xi|\geq\lambda} \langle x\rangle^{-2} \left(\tau^2 + |\xi|^2\right),$$

where $q = \tau q_0 + q_1$.

Here, $S^m(T^*\mathbb{R}^n)$ denotes the m'th order *Kohn-Nirenberg* symbol class

$$S^m(T^*\mathbb{R}^n) = \left\{p \in C^\infty(T^*\mathbb{R}^n) : |D_x^\alpha D_\xi^\beta p(x,\xi)| \lesssim_{\alpha,\beta} \langle\xi\rangle^{m-|\beta|}\right\}.$$

To each symbol $p(x,\xi) \in S^m(T^*\mathbb{R}^n)$, we will associate the pseudodifferential operator $p^{\mathrm{w}}(x,D) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, namely the *Weyl quantization* of p. This is defined via the action

$$p^{\mathrm{w}}(x,D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} p\left(\frac{x+y}{2},\xi\right) u(y)\, dyd\xi.$$

We say that the Weyl quantization of $p \in S^m(T^*\mathbb{R}^n)$ is an element of the class $\Psi^m(\mathbb{R}^n)$, i.e. $\Psi^m(\mathbb{R}^n) = \{p^{\mathrm{w}}(x,D) : p \in S^m(T^*\mathbb{R}^n)\}$. For more on pseudodifferential operators and their corresponding calculus, we refer the reader to [12], [46], [49] (the first two for the microlocal framework, the latter for the semiclassical framework).

In the proof of Lemma 3.4, it will be useful to work with the half-wave decomposition (which allows us to avoid the cross terms in the principal symbol). To that end, we factor p as

$$p(\tau, x, \xi) = (\tau - b^+(x,\xi))(\tau - b^-(x,\xi)),$$

where

$$b^\pm(x,\xi) = g^{0j}\xi_j \pm \sqrt{(g^{0j}\xi_j)^2 + g^{ij}\xi_i\xi_j}.$$

Observe that $b^\pm$ are both homogeneous of degree 1 in ξ. Additionally, they are both signed.

Proposition 3.5. *For any $(x,\xi) \in T^*\mathbb{R}^3 \setminus o$, we have that $b^+(x,\xi) > 0 > b^-(x,\xi)$.*

Proof. Let $\xi \neq 0$. First, we show that $b^+ > b^-$. Indeed, observe that

$$b^+ - b^- = 2\sqrt{(g^{0j}\xi_j)^2 + g^{ij}\xi_i\xi_j} > 0$$

using the ellipticity (see 1.2). Using ellipticity again, we have that

$$\sqrt{(g^{0j}\xi_j)^2 + g^{ij}\xi_i\xi_j} > |g^{0j}\xi_j|.$$

Thus,

$$b^+ > g^{0j}\xi_j + |g^{0j}\xi_j| \geq 0, \qquad b^- < g^{0j}\xi_j - |g^{0j}\xi_j| \leq 0.$$

□

We will call $p^\pm = \tau - b^\pm$, so that $p = p^+p^-$. In particular, $p = 0$ if and only if $p^+ = 0$ or $p^- = 0$; due to Proposition 3.5, it cannot be the case that $p^+(w) = p^-(w) = 0$ for any $w \in T^*\mathbb{R}^4 \setminus o$. The Hamiltonians $p^\pm$ also generate flows $\varphi_s^\pm(w) = (t_s^\pm(w), \tau_s^\pm(w), x_s^\pm(w), \xi_s^\pm(w))$ on $\mathbb{R} \times T^*\mathbb{R}^4$ which solve the Hamiltonian systems

$$\begin{cases} \dot{t}_s^\pm = \partial_\tau p^\pm(\varphi_s^\pm(w)), \\ \dot{\tau}_s^\pm = -\partial_t p^\pm(\varphi_s^\pm(w)), \\ \dot{x}_s^\pm = \nabla_\xi p^\pm(\varphi_s^\pm(w)), \\ \dot{\xi}_s^\pm = -\nabla_x p^\pm(\varphi_s^\pm(w)) \end{cases}$$

with initial data $w \in T^*\mathbb{R}^4$. Note that

$$\begin{cases} \dot{t}_s^\pm = 1, \\ \dot{\tau}_s^\pm = 0, \\ (\dot{x}_s^\pm)_k = -b^\pm_{\xi_k}(\varphi_s^\pm(w)), \\ (\dot{\xi}_s^\pm)_k = b^\pm_{x_k}(\varphi_s^\pm(w)). \end{cases}$$

There is a direct correspondence between null bicharacteristics for φ_s and null bicharacteristics for $\varphi_s^\pm$.

Proposition 3.6. *Every null bicharacteristic for the flow generated by p is a null bicharacteristic for the flow generated by either p^+ or p^-. The converse is also true.*

Proof. Recall that for any $(t', \tau', x', \xi') =: w \in T^*\mathbb{R}^4 \setminus o$, $p(w) = 0$ if and only if either $p^+(w) = 0$ or $p^-(w) = 0$. Without loss of generality, suppose that $p^+(w) = 0$. The Hamiltonians p and p^+ generate the systems

$$\begin{cases} \dot{t}_s = p^+(\varphi_s) + p^-(\varphi_s), \\ \dot{\tau}_s = 0, \\ (\dot{x}_s)_k = p^+(\varphi_s)p^-_{\xi_k}(\varphi_s) + p^-(\varphi_s)p^+_{\xi_k}(\varphi_s), \\ (\dot{\xi}_s)_k = -p^+(\varphi_s)p^-_{x_k}(\varphi_s) - p^-(\varphi_s)p^+_{x_k}(\varphi_s), \\ (t_s, \tau_s, x_s, \xi_s)\big|_{s=0} = w \end{cases} \tag{3.2}$$

and

$$\begin{cases} \dot{t}_s^+ = 1, \\ \dot{\tau}_s^+ = 0, \\ (\dot{x}_s^+)_k = p^+_{\xi_k}(\varphi_s^+), \\ (\dot{\xi}_s^+)_k = -p^+_{x_k}(\varphi_s^+), \\ (t_s^+, \tau_s^+, x_s^+, \xi_s^+)\big|_{s=0} = w, \end{cases} \tag{3.3}$$

respectively.

We claim that since $p^+(w) = 0$, we must have that $p^+(\varphi_s(w)) = 0$ for all s. If not, then there would exist s' so that $p^-(\varphi_{s'}(w)) = 0$, i.e. $\tau^-_{s'}(w) = b^-(x_{s'}(w), \xi_{s'}(w)) < 0$. However, τ^- is constant and $p^+(w) = 0$, which implies that $\tau^-_s(w) = \tau' > 0$ for all s.

Thus, we can re-write (3.2) as

$$\text{(3.4)} \qquad \begin{cases} \dot{t}_s = p^-(\varphi_s), \\ \dot{\tau}_s = 0, \\ (\dot{x}_s)_k = p^-(\varphi_s) p^+_{\xi_k}(\varphi_s), \\ (\dot{\xi}_s)_k = -p^-(\varphi_s) p^+_{x_k}(\varphi_s), \\ (t_s, \tau_s, x_s, \xi_s)\big|_{s=0} = w. \end{cases}$$

Notice that $t^+ = t' + s$, and so we may re-parameterize (3.3) in terms of t^+:

$$\text{(3.5)} \qquad \begin{cases} \dfrac{d}{dt^+} t^+_{t^+ - t'} = 1, \\ \dfrac{d}{dt^+} \tau^+_{t^+ - t'} = 0, \\ \left(\dfrac{d}{dt^+} x^+_{t^+ - t'} \right)_k = p^+_{\xi_k}(\varphi^+_{t^+ - t'}), \\ \left(\dfrac{d}{dt^+} \xi^+_{t^+ - t'} \right)_k = -p^+_{x_k}(\varphi^+_{t^+ - t'}), \\ \left(t^+_{t^+ - t'}, \tau^+_{t^+ - t'}, x^+_{t^+ - t'}, \xi^+_{t^+ - t'} \right) \big|_{t^+ = t'} = w. \end{cases}$$

Next, we re-parameterize (3.4) to change the flow variable from s to t (which can be done since t_s is strictly increasing and hence invertible), generating the system

$$\begin{cases} \dfrac{d}{dt} t_{s(t)} = 1, \\ \dfrac{d}{dt} \tau_{s(t)} = 0, \\ \left(\dfrac{d}{dt} x_{s(t)} \right)_k = p^+_{\xi_k}(\varphi_{s(t)}), \\ \left(\dfrac{d}{dt} \xi_{s(t)} \right)_k = -p^+_{x_k}(\varphi_{s(t)}), \\ (t_{s(t)}, \tau_{s(t)}, x_{s(t)}, \xi_{s(t)})\big|_{t=t'} = w. \end{cases}$$

An application of uniqueness theory yields that $\varphi_{s(t)}(w) = \varphi^+_{t^+ - t'}(w)$. The converse is similar by reversing the above process. $\square$

When working with the factored flow, the decoupling of (t,τ) and (x,ξ) allows us to project onto the (x,ξ) components of the flow without worrying about loss of information. For this reason, we will write $\Pi_{x,\xi} \circ \varphi^{\pm}$ as simply $\varphi^{\pm}$, where $\Pi_{x,\xi}(t,\tau,x,\xi) = (x,\xi)$. Notice that when we project, we are no longer looking at null bicharacteristics but, rather, bicharacteristics with initial data having non-zero ξ component.

Now, we may define all of the corresponding trapped and non-trapped sets for the half-wave flows as

$$\begin{aligned}
\Gamma^{\pm}_{tr} &= \left\{ w \in T^*\mathbb{R}^3 \setminus o : \sup_{s \geq 0} |x^{\pm}_s(w)| < \infty \right\}, \\
\Lambda^{\pm}_{tr} &= \left\{ w \in T^*\mathbb{R}^3 \setminus o : \sup_{s \geq 0} |x^{\pm}_{-s}(w)| < \infty \right\}, \\
\Omega^{\pm}_{tr} &= \Gamma^{\pm}_{tr} \cap \Lambda^{\pm}_{tr}, \\
\Omega_{tr} &= \Omega^{+}_{tr} \cup \Omega^{-}_{tr}, \\
\Gamma^{\pm}_{\infty} &= \left\{ w \in T^*\mathbb{R}^3 \setminus o : |x^{\pm}_s(w)| \to \infty \ \text{ as } \ s \to \infty \right\}, \\
\Lambda^{\pm}_{\infty} &= \left\{ w \in T^*\mathbb{R}^3 \setminus o : |x^{\pm}_{-s}(w)| \to \infty \ \text{ as } \ s \to \infty \right\}, \\
\Omega^{\pm}_{\infty} &= \Gamma^{\pm}_{\infty} \cap \Lambda^{\pm}_{\infty}, \\
\Omega_{\infty} &= \Omega^{+}_{\infty} \cup \Omega^{-}_{\infty}.
\end{aligned}$$

Note that the identities

$$\Omega_{tr} = \Pi_{x,\xi}(\Omega^{p}_{tr}), \qquad \Omega_{\infty} = \Pi_{x,\xi}(\Omega^{p}_{\infty})$$

hold as an immediate consequence of the factoring. Additionally, the factoring allows us to restate the geometric control condition as

$$\left(w \in \Omega^{+}_{tr} \implies (\exists s \in \mathbb{R}) \ \left(a(x^{+}_s(w)) > 0 \right) \right) \ \text{ and } \ \left(w \in \Omega^{-}_{tr} \implies (\exists s \in \mathbb{R}) \ \left(a(x^{-}_s(w)) > 0 \right) \right).$$

If $w \in \Omega_{tr}$, then it is either trapped with respect the flow generated by p^{+} or p^{-} by Proposition 3.6. If it is trapped with respect to p^+, then there is a time so that w is flowed along a p^+-bicharacteristic ray to a place where the damping is positive, and similarly if it is trapped with respect to p^-.

3.3 Results on the Flow

Here, we establish results regarding the trapped/non-trapped sets and scalings for the flows, culminating in an extension of geometric control to bicharacteristic rays bounded either forward or backward in time. These results largely follow the path outlined in [6], although we require certain scaling results in order to utilize homogeneity arguments in later proofs (which were unnecessary in [6] due to their use of semiclassical rescaling). In particular, Lemma 3.10 and Propositions 3.11 and 3.12 are analogous to results in Chapter 8 of [6].

We will start with a scaling result on the flow.

Proposition 3.7. *The flows generated by $p^\pm$ satisfy the scalings*

$$\begin{aligned} x_s^\pm(x,\xi) &= x_s^\pm(x,\lambda\xi), \\ \lambda\xi_s^\pm(x,\xi) &= \xi_s^\pm(x,\lambda\xi) \end{aligned}$$

for any $\lambda > 0$.

Proof. Label the functions on the right-hand side as $x_{s,\lambda}^\pm$ and $\xi_{s,\lambda}^\pm$, respectively. Using the homogeneity of $b^\pm$, the left-hand side $(x_s^\pm, \lambda\xi_s^\pm)$ solves the system

$$\begin{cases} \dfrac{d}{ds}x_s^\pm = \nabla_\xi p^\pm(x_s^\pm, \xi_s^\pm) = \nabla_\xi p^\pm(x_s^\pm, \lambda\xi_s^\pm), \\ \dfrac{d}{ds}(\lambda\xi_s^\pm) = -\lambda\nabla_x p^\pm(x_s^\pm, \xi_s^\pm) = -\nabla_x p^\pm(x_s^\pm, \lambda\xi_s^\pm), \\ (x_s^\pm, \lambda\xi_s^\pm)\big|_{s=0} = (x, \lambda\xi), \end{cases}$$

while the right-hand side solves

$$\begin{cases} \dfrac{d}{ds}x_{s,\lambda}^\pm = \nabla_\xi p^\pm(x_{s,\lambda}^\pm, \xi_{s,\lambda}^\pm), \\ \dfrac{d}{ds}(\xi_{s,\lambda}^\pm) = -\nabla_x p^\pm(x_{s,\lambda}^\pm, \xi_{s,\lambda}^\pm), \\ (x_{s,\lambda}^\pm, \xi_{s,\lambda}^\pm)\big|_{s=0} = (x, \lambda\xi). \end{cases}$$

In particular, $(x_s^\pm, \lambda\xi_s^\pm)$ and $(x_{s,\lambda}^\pm, \xi_{s,\lambda}^\pm)$ solve the same ordinary differential equations with the same initial conditions; applying uniqueness theory completes the proof. □

This scaling implies that the trapped/non-trapped sets, and hence geometric control, are entirely determined by unit speed null bicharacteristics, i.e. by what happens on the unit cosphere bundle $S^*\mathbb{R}^3 = \{(x,\xi) \in T^*\mathbb{R}^3 : |\xi| = 1\}$. Indeed, observe that

$$x_s^\pm(x,\xi) = x_s^\pm(x, \xi/|\xi|).$$

The forward/backward trapped sets are defined in terms of supremums of the above over s, while the forward/backward non-trapped sets are defined via limits in s, and the prior equation shows that all of these are unaffected by the scaling in the ξ component of the initial data. A more pertinent scaling is given by the function

$$\Phi^\pm(x,\xi) = \left(x, \frac{\xi}{|b^\pm(x,\xi)|}\right).$$

The utility of this scaling comes from noticing that $b^\pm$ is a constant of motion under the corresponding projected Hamiltonian flows and that

$$\left|\frac{\xi}{b^+(x,\xi)}\right| \approx 1,$$

which we now prove.

Proposition 3.8. *For any* $(x,\xi) \in T^*\mathbb{R}^3 \setminus o$,

$$\left|\frac{\xi}{b^\pm(x,\xi)}\right| \approx 1.$$

Proof. By homogeneity,

$$\left|\frac{\xi}{b^\pm(x,\xi)}\right| = \frac{1}{\left|b^\pm\left(x, \frac{\xi}{|\xi|}\right)\right|}.$$

Write

$$\left|b^\pm\left(x, \frac{\xi}{|\xi|}\right)\right| = \left|g^{0j}\frac{\xi_j}{|\xi|} \pm \sqrt{\left(g^{0j}\frac{\xi_j}{|\xi|}\right)^2 + (g^{ij} - m^{ij})\frac{\xi_i}{|\xi|}\frac{\xi_j}{|\xi|} + m^{ij}\frac{\xi_i}{|\xi|}\frac{\xi_j}{|\xi|}}\right|.$$

Since $\|g - m\|_{AF(|x|>R_0)} \ll 1$, asymptotic flatness guarantees that g^{0j} and $g^{ij} - m^{ij}$ are small in the exterior region $\{|x| > R_0\}$. Hence,

$$\left| b^{\pm}\left(x, \frac{\xi}{|\xi|}\right)\right| \approx \sqrt{m^{ij}\frac{\xi_i}{|\xi|}\frac{\xi_j}{|\xi|}} = 1$$

when $|x| > R_0$.

In the interior region, we are considering $b^{\pm}$ on the compact set $\{|x| \leq R_0\} \times \{|\xi| = 1\}$. Since we know that $|b^{\pm}| > 0$ for all $\xi \neq 0$ from Proposition 3.5, continuity guarantees the desired boundedness here. □

In view of Proposition 3.8, it follows that

$$\frac{\xi_j}{b^{\pm}(x,\xi)} \in S^0_{\text{hom}}(T^*\mathbb{R}^3 \setminus o), \qquad j = 1, 2, 3,$$

where $S^0_{\text{hom}}(T^*\mathbb{R}^3 \setminus o)$ denotes the 0'th-order homogeneous symbol class.

Remark 3.9. As a consequence of the scaling, the sets

$$\dot{\Gamma}^{\pm}_{tr} = \Gamma^{\pm}_{tr} \cap \Phi^{\pm}(T^*\mathbb{R}^3 \setminus o)$$
$$\dot{\Lambda}^{\pm}_{tr} = \Lambda^{\pm}_{tr} \cap \Phi^{\pm}(T^*\mathbb{R}^3 \setminus o)$$

are invariant under the flow. Indeed, it is readily seen that the (semi) trapped nature is preserved. Further, since $b^{\pm}$ is constant along the flow, it follows from Proposition 3.7 that

$$\xi^{\pm}_s\left(x, \frac{\xi}{|b^{\pm}(x,\xi)|}\right) = \frac{1}{|b^{\pm}(x,\xi)|}\xi^{\pm}_s(x,\xi) = \frac{1}{|b^{\pm}(x^{\pm}_s(x,\xi), \xi^{\pm}_s(x,\xi))|}\xi^{\pm}_s(x,\xi).$$

■

Now, we prove a key result on non-trapped trajectories.

Lemma 3.10. *If $R \geq R_0$ and*

$$|x^{\pm}_{\pm s'}(x,\xi)| \geq \max\{2R, |x| + \delta\}$$

for some $(x,\xi) \in T^*\mathbb{R}^3 \setminus o$, $\delta > 0$, and $s' > 0$, then it holds for all $s \geq s'$, and

$$|x^{\pm}_{\pm s}(x,\xi)| \to \infty$$

as $s \to \infty$.

That is, if we can get sufficiently far away from the origin and move radially outward from the initial position, then the trajectories are necessarily non-trapped. This can be proven directly, but the computations are simpler if one uses the correspondence between null bicharacteristics for p and $p^{\pm}$.

Proof. Without loss of generality, we will work with the x^+ bicharacteristic ray. By Proposition 3.6, it suffices to prove the result for the null bicharacteristic ray $x_{\pm s}$ with initial data $\tilde{w}$, where $\tilde{w}$ is the lift of w to $T^*\mathbb{R}^4 \setminus o$ which is consistent with the comment immediately following the aforementioned proposition (in particular, the τ component is strictly positive). For any $z \in T^*\mathbb{R}^4 \setminus o$, we explicitly calculate that

$$\frac{\partial^2}{\partial s^2}|x_{\pm s}(z)|^2 = \left|\frac{\partial}{\partial s}x_{\pm s}(z)\right|^2 + x_{\pm s}(z) \cdot \frac{\partial^2}{\partial s^2}x_{\pm s}(z),$$

where

$$\begin{aligned}\left|\frac{\partial}{\partial s}x_{\pm s}(z)\right|^2 = 4\tau^2_{\pm s}(z)\left(\sum_{k=1}^{3} g^{0k}(x_{\pm s}(z))\right)^2 &+ 4\sum_{k=1}^{3}[(g^{ki}(x_{\pm s}(z))(\xi_{\pm s}(z))_i][(g^{kj}(x_{\pm s}(z))(\xi_{\pm s}(z))_j] \\ &+ 8\sum_{k=1}^{3}\tau_{\pm s}(z)g^{0k}(x_{\pm s}(z))g^{kj}(x_{\pm s}(z))(\xi_j(z))_{\pm s},\end{aligned}$$

and

$$
\begin{aligned}
&x_{\pm s}(z) \cdot \frac{\partial^2}{\partial s^2} x_{\pm s}(z) \\
&= 4\tau_{\pm s}(z)(x_{\pm s}(z))_k[\partial_\ell g^{0k}(x_{\pm s}(z))] \left(\tau_{\pm s}(z) g^{0\ell}(x_{\pm s}(z)) + g^{\ell j}(x_{\pm s}(z))(\xi_{\pm s}(z))_j\right) \\
&\quad + 4(x_{\pm s}(z))_j[\partial_\ell g^{kj}(x_{\pm s}(z))] \left(\tau_{\pm s}(z) g^{0\ell}(x_{\pm s}(z)) + g^{\ell j}(x_{\pm s}(z))(\xi_{\pm s}(z))_j\right)(\xi_{\pm s}(z))_j \\
&\quad - 2(x_{\pm s}(z))_k g^{kj}(x_{\pm s}(z)) \left(2\tau_{\pm s}(z)\partial_j g^{0i}(x_{\pm s}(z))(\xi_{\pm s}(z))_i + \partial_j g^{i\ell}(x_{\pm s}(z))(\xi_{\pm s}(z))_i(\xi_{\pm s}(z))_\ell\right).
\end{aligned}
$$

Since τ_s is constant for stationary metrics, it follows that

$$
\tau_{\pm s}(z) = \tau_0 = b^+(\Pi_{x,\xi}(z)) = b^\pm\left(x^+_{\pm s}(\Pi_{x,\xi}(z)), \xi^+_{\pm s}(\Pi_{x,\xi}(z))\right) \approx |\xi^+_{\pm s}(\Pi_{x,\xi}(z))|,
$$

and so

$$
\frac{\partial^2}{\partial s^2}|x_{\pm s}(z)|^2 \gtrsim |\xi^+_{\pm s}(\Pi_{x,\xi}(z))|^2\left(1 - \|g-m\|_{AF_{>R}}\right)
$$

provided that $|x_{\pm s}(z)| > R$. In such a case, we have that $\|g-m\|_{AF_{>R}} \ll 1$, and thus

$$
\frac{\partial^2}{\partial s^2}|x_{\pm s}(z)|^2 > 0.
$$

By a mean value theorem argument, there exists $s'' \in [0, s']$ such that

$$
\begin{gathered}
|x_{\pm s''}(\tilde{w})|^2 > R^2 \\
\left(\frac{\partial}{\partial s}|x_{\pm s}(\tilde{w})|^2\right)\Big|_{s=s''} > 0.
\end{gathered}
$$

All together, we have that $|x_{\pm s}(\tilde{w})|^2$ has positive derivative at $s = s''$, and its derivative is increasing for all $s \geq s''$. In particular, $|x_{\pm s}(\tilde{w})|^2$ is increasing for all $s \geq s''$, which implies the result. □

As a consequence, we can use the trapped and non-trapped sets to partition phase space.

Proposition 3.11.

(a) *We can partition* $T^*\mathbb{R}^3 \setminus o$ *as*

$$T^*\mathbb{R}^3 \setminus o = \Gamma^\pm_{tr} \sqcup \Gamma^\pm_\infty = \Lambda^\pm_{tr} \sqcup \Lambda^\pm_\infty,$$
$$T^*\mathbb{R}^3 \setminus o = \Gamma^\pm_{tr} \cup \Lambda^\pm_{tr} \cup \Omega^\pm_\infty.$$

(b) $\Gamma^\pm_\infty, \Lambda^\pm_\infty, \Omega^\pm_\infty$ *are open in* $T^*\mathbb{R}^3 \setminus o$*, and* $\Gamma^\pm_{tr}, \Lambda^\pm_{tr}, \Omega^\pm_{tr}$ *are closed.*

(c) *If* $K \subset \Omega^\pm_\infty$ *is compact, then for every* $R \geq R_0$*, there exists* $T' \geq 0$ *so that*

$$|x^\pm_s(v)| > R$$

for every $|s| \geq T'$ *and* $v \in K$*. Also,*

$$\bigcup_{s\in\mathbb{R}} \varphi^\pm_s(K)$$

is closed in $T^*\mathbb{R}^3 \setminus o$*.*

Proof. Without loss of generality, we will only work with the flow generated by p^+. Let $R \geq R_0$.

(a) It is readily seen that

$$\Gamma^+_{tr} \cap \Gamma^+_\infty = \Lambda^+_{tr} \cap \Lambda^+_\infty = \emptyset.$$

Let $(x, \xi) := w \in (T^*\mathbb{R}^3 \setminus o) \setminus \Gamma^+_{tr}$. Then, there exists $s_w > 0$ such that

$$|x^+_{s_w}(w)| \geq \max\{2R, |x| + 1\}.$$

By Lemma 3.10,

$$|x^+_s(w)| \to \infty \text{ as } s \to \infty,$$

which implies that $(x, \xi) \in \Gamma^+_\infty$. This proves the first equality. Proving that

$$T^*\mathbb{R}^3 \setminus o = \Lambda^+_{tr} \sqcup \Lambda^+_\infty$$

is similar, and the fact that

$$T^*\mathbb{R}^3 \setminus o = \Gamma^+_{tr} \cup \Lambda^+_{tr} \cup \Omega^+_\infty$$

now follows immediately.

(b) We only show that Γ^+_∞ is open, as the rest either follow similarly or by taking complements and using part (a). Given any $(x,\xi) := w \in \Gamma^+_\infty$, there exists $s_w > 0$ such that

$$|x^+_{s_w}(w)| \geq 2\max\{2R, |x|+1\}.$$

By continuous dependence of the flow on the data, $\delta > 0$ exists so that

$$|w' - w| < \delta \implies |x^+_{s_w}(w') - x^+_{s_w}(w)| < \frac{1}{2}|x^+_{s_w}(w)|.$$

That is, if $|w'-w| < \delta$, then

$$|x^+_{s_w}(w')| > \frac{1}{2}|x^+_{s_w}(w)| \geq \max\{2R, |x|+1\}.$$

By Lemma 3.10, $B_\delta(w) \subset \Gamma^+_\infty$.

(c) Let $(x,\xi) =: w \in K \subset \Omega^+_\infty$. As in the proof of part (b), we can find a time $s_w > 0$ and an open neighborhood U_w of w such that

$$|x^+_{\pm s_w}(w')| \geq \max\{2R, |x|+1\} > R$$

for all $w' \in U_w$. By Lemma 3.10, this holds for all $s \geq s_w$. Now, we cover the compact set K with neighborhoods $\{U_w\}_{w\in K}$, which can be reduced to a finite subcover $\{U_{w_j}\}_{j=1}^N$, with $w_j \in K$ for $j = 1, 2, \cdots, N$. Calling $T' = \max_{1\leq j\leq N} s_{w_j}$ completes the proof of the first claim.

For the second claim, observe that the first claim in part (c) implies that for any $R \geq R_0$,

$$\bigcup_{s\in\mathbb{R}} \varphi^\pm_s(K) \cap \{|x| \leq R\} = \bigcup_{s\in[-T',T']} \varphi^\pm_s(K) \cap \{|x| \leq R\} = \varphi^\pm([-T',T'] \times K) \cap \{|x| \leq R\}.$$

Since $[-T', T']$ and K are compact and $\varphi_s^{\pm}$ is continuous in the flow parameter and depends continuously on the data, it follows that the above set is a compact set.

In order to demonstrate that $\bigcup_{s\in\mathbb{R}} \varphi_s^{\pm}(K)$ is closed, we take a sequence (w_n) in $\bigcup_{s\in\mathbb{R}} \varphi_s^{\pm}(K)$ which converges to some $w \in T^*\mathbb{R}^3 \setminus o$. Say that $w_n = (x_n, \xi_n)$, and $w = (x, \xi)$. Since w_n converges, there exists $R \geq R_0$ so that

$$w_n \in \bigcup_{s\in[-T',T']} \varphi_s^{\pm}(K) \cap \{|x| \leq R\}.$$

Since this set is closed, it follows that

$$w \in \bigcup_{s\in[-T',T']} \varphi_s^{\pm}(K) \cap \{|x| \leq R\} = \bigcup_{s\in\mathbb{R}} \varphi_s^{\pm}(K) \cap \{|x| \leq R\} \subset \bigcup_{s\in\mathbb{R}} \varphi_s^{\pm}(K).$$

□

Finally, we show that if one assumes geometric control for bounded bicharacteristic rays, then it holds for semi-bounded bicharacteristic rays (that is, those which are bounded forward or backward in time).

Proposition 3.12. *Assume that the geometric control condition (3.1) holds. If $w \in \dot{\Gamma}_{tr}^{\pm}$, then there exists $s_{\pm} \geq 0$ so that $a\left(x_{s_{\pm}}^{\pm}(w)\right) > 0$. The same is true for $w \in \dot{\Lambda}_{tr}^{\pm}$, but with $s_{\pm} \leq 0$.*

Proof. We will only demonstrate this for $\dot{\Gamma}_{tr}^{+}$, as the work to establish the remaining cases is similar. If $w \in \dot{\Gamma}_{tr}^{+}$, then

$$\alpha := \sup_{s\geq 0} |x_s^{+}(w)| < \infty.$$

According to Remark 3.9, $|\xi_s^{+}(w)| \approx 1$ for all $s \in \mathbb{R}$. Thus,

$$w' := \sup_{s\geq 0} |\varphi_s^{+}(w)| < \infty.$$

Then, there exists a sequence (s_n) of non-negative real numbers such that $\varphi_{s_n}^{+}(w) \to w'$ as $s_n \to \infty$. For any $s \in \mathbb{R}$, the group law for the flow tells us that

$$\varphi_{s+s_n}^{+}(w) = \varphi_s^{+}(\varphi_{s_n}^{+}(w)),$$

and so

$$x^+_{s+s_n}(w) = \Pi_x \circ \varphi^+_s(\varphi^+_{s_n}(w)) \to x^+_s(w') \text{ as } s_n \to \infty.$$

Since $s + s_n \geq 0$ for large enough n, it follows that $|x^+_s(w')| \leq \alpha$ for all $s \in \mathbb{R}$. By (3.1), there exists $s' \in \mathbb{R}$ for which $a(x^+_{s'}(w')) > 0$. Recall that $x^+_{s'+s_n}(w) \to x^+_{s'}(w')$ as $n \to \infty$. Since a is continuous and $s' + s_n \geq 0$ for n large enough, we conclude that

$$a\left(x^+_{s'+s_N}(w)\right) > 0$$

for some large N. □

3.4 Escape Function Construction

We will construct our symbols in multiple steps:

1. **On the characteristic set.** Since we are utilizing the half-wave decomposition, working on the characteristic set amounts to working on each individual light cone, then combining together. There are three regions of interest, two sub-regions of the *interior region* $\{|x| \leq R\}$ and the *exterior region* $\{|x| > R\}$. Here, $R \geq R_0$.

   (a) **Interior, semi-bounded null bicharacteristics.** As opposed to working with the trapped and non-trapped sets, we will first work with the semi-bounded null bicharacteristics with initial data living in the interior region $\{|x| \leq R\}$. Working with the trapped and non-trapped sets can be difficult, since one can have non-trapped trajectories which are bounded forward or backward in time (but not both). Heuristically, these trajectories constitute the boundary of the non-trapped set. Instead, we will explicitly work with trajectories which are bounded forward or backward in time. This is where geometric control is used. This step is inspired by the work in [6].

   (b) **The remainder of the interior region.** Since there is no trapping here, we construct a symbol similar to the one constructed in [6], [11], and [27]. We will need to make an appropriate modification to avoid trapped trajectories while working with the half-wave symbols.

(c) **The exterior region.** As a consequence of asymptotic flatness, there are no trapped trajectories here. Hence, this follows from a similar multiplier to that used to prove local energy decay for the flat wave equation, although the multiplier must be appropriately adapted to the geometry. Here, we are motivated by prior work in [23] and [27].

2. **On the elliptic set.** Here, we construct a correction term. That is, we will construct a lower-order symbol which provides no contribution on the characteristic set and provides positivity off of it. This is based on the work in [27].

We will break this construction up into a sequence of lemmas, starting with (1a). While our construction follows that of [6], we reason differently. Their argument utilizes semiclassical rescaling, which provides compactness for their interior, semi-trapped set. Since we are sticking with the microlocal framework, we instead utilize homogeneity arguments to obtain this compactness. This is one of the reasons to work with the half-wave decomposition (the other being related to step (1b), which we will outline once we get there).

With this in mind, we will utilize the sets

$$\Omega_R^{\pm} := \left(\Gamma_{tr}^{\pm} \cup \Lambda_{tr}^{\pm}\right) \cap \{|x| \leq R\},$$
$$\dot{\Omega}_R^{\pm} := \Omega_R^{\pm} \cap \Phi^{\pm}(T^*\mathbb{R}^3 \setminus o).$$

As a consequence of Proposition 3.8 and Proposition 3.11(b), the latter set is compact.

Lemma 3.13 (Semi-bounded Escape Function Construction)**.** *There exist $q^{\pm} \in C^{\infty}(T^*\mathbb{R}^3 \setminus o)$, an open set $V_R^{\pm} \supset \Omega_R^{\pm}$, and $C^{\pm} \in \mathbb{R}_+$ so that*

$$H_{p^{\pm}} q^{\pm} + C^{\pm} a \gtrsim_R \mathbb{1}_{V_R^{\pm}}.$$

Further, $q^{\pm} = q_1^{\pm} \circ \Phi^{\pm}$, where $q_1^{\pm} \in C_c^{\infty}(T^*\mathbb{R}^3 \setminus o)$.

Here, $\Phi \in S^0_{\text{hom}}(T^*\mathbb{R}^3 \setminus o)$ is the scaling function introduced in Section 3.3. The fact that we omit the zero section is unavoidable, but it is non-problematic; we will introduce high-frequency cutoffs to our symbols later on which allow for smooth extensions to all of phase space.

Proof. We will first construct a symbol $q_1^\pm$ and an open set $\dot{V}_R^\pm \supset \dot{\Omega}_R^\pm$ such that

$$H_{p^\pm} q_1^\pm + C^\pm a \gtrsim_R \mathbb{1}_{\dot{V}_R^\pm}.$$

To that end, let $w^\pm \in \dot{\Omega}_R^\pm$. By Proposition 3.12, there exists $s_{w^\pm} \in \mathbb{R}$ for which $a(x^\pm_{s_{w^\pm}}(w^\pm)) > 0$. Say that $2\alpha_{w^\pm} := a(x^\pm_{s_{w^\pm}}(w^\pm))$. By the continuity of the flow in the initial data, there exists a neighborhood $U_{w^\pm}$ of $w^\pm$ so that $a(x^\pm_{s_{w^\pm}}(z)) > \alpha_{w^\pm}$ for all $z \in U_{w^\pm}$. Select a smooth cutoff $\chi_{w^\pm} \in C_c^\infty(T^*\mathbb{R}^3)$ so that supp $\chi_{w^\pm} \subset U_{w^\pm}$ and $\chi_{w^\pm} \equiv 1$ on a smaller neighborhood $V_{w^\pm}$ of $w^\pm$. Now, we define a symbol on $T^*\mathbb{R}^3 \setminus o$ given by

$$q_{w^\pm}(x,\xi) = \int_0^{s_{w^\pm}} \left(\chi_{w^\pm} \circ \varphi^\pm_{-s}\right)(x,\xi)\, ds.$$

Such a symbol is readily seen to be well-defined, and it is smooth by the aforementioned smooth flow dependence on data. Next, we demonstrate its symbolic nature. By continuity of the flow, $\varphi^\pm_{[0,s_{w^\pm}]}(\overline{U_{w^\pm}}) := \varphi^\pm([0,s_{w^\pm}] \times \overline{U_{w^\pm}})$ is compact. If $(x,\xi) \notin \varphi^\pm_{[0,s_{w^\pm}]}(\overline{U_{w^\pm}})$, then $(x,\xi) \notin \varphi^\pm_s(\overline{U_{w^\pm}})$ for any $s \in [0, s_w]$. Then, $\varphi^\pm_{-s}(x,\xi) \notin \overline{U_{w^\pm}}$ for any $s \in [0, s_{w^\pm}]$, implying that $q_{w^\pm}(x,\xi) = 0$. Hence, $q_{w^\pm} \in C_c^\infty(T^*\mathbb{R}^3 \setminus o)$.

Applying the Hamiltonian vector field $H_{p^\pm}$ gives us

$$H_{p^\pm} q_{w^\pm} = \int_0^{s_{w^\pm}} H_{p^\pm}(\chi_{w^\pm} \circ \varphi^\pm_{-s})\, ds = -\int_0^{s_{w^\pm}} \partial_s \left(\chi_{w^\pm} \circ \varphi^\pm_{-s}\right)\, ds = \chi_{w^\pm} - \chi_{w^\pm} \circ \varphi^\pm_{-s_{w^\pm}}.$$

Notice that the term $-\chi_{w^\pm} \circ \varphi^\pm_{-s_{w^\pm}}$ is non-positive and that

$$\text{supp}\left(\chi_{w^\pm} \circ \varphi^\pm_{-s_{w^\pm}}\right) \subset \left\{v : \varphi^\pm_{-s_{w^\pm}}(v) \in U_{w^\pm}\right\} = \left\{v : v \in \varphi^\pm_{s_{w^\pm}}(U_{w^\pm})\right\} \subset \{x : a(x) > \alpha_{w^\pm}\}.$$

Using this support property, we can use the damping to absorb the poorly-signed term and obtain non-negativity of $H_{p^\pm} q_{w^\pm}$. Indeed, if we call $C_{w^\pm} = 2(\alpha_{w^\pm})^{-1}$, then we have

$$\chi_{w^\pm} \circ \varphi_{-s_{w^\pm}} + C_{w^\pm} a(x) \geq 0.$$

Thus,

$$H_{p^\pm} q_{w^\pm} + C_{w^\pm} a \gtrsim \mathbb{1}_{V_{w^\pm}}.$$

Since $\dot{\Omega}_R^\pm$ is compact, we can reduce the open cover $\{V_{w^\pm}\}_{w^\pm \in \dot{\Omega}_R^\pm}$ to a finite subcover $\{V_{w_j^\pm}\}_{j=1}^m$, with each $w_j^\pm \in \dot{\Omega}_R^\pm$. Call

$$\dot{V}_R^\pm = \bigcup_{j=1}^m V_{w_j^\pm}, \qquad q_1^\pm = \sum_{j=1}^m q_{w_j^\pm}, \qquad \text{and} \qquad C^\pm = \sum_{j=1}^m C_{w_j^\pm}.$$

This provides us with a symbol $q_1^\pm \in C_c^\infty(T^*\mathbb{R}^3 \setminus o)$ so that

$$H_{p^\pm} q_1^\pm + C^\pm a \gtrsim \mathbb{1}_{\dot{V}_R^\pm}, \qquad \dot{V}_R^\pm \supset \dot{\Omega}_R^\pm.$$

Finally, we will extend the above estimate from an indicator on $\dot{V}_R^\pm$ to an indicator on a neighborhood $V_R^\pm \supset \Omega_R^\pm$. Consider the function $q^\pm : T^*\mathbb{R}^3 \setminus o \to \mathbb{R}$ given by

$$q^\pm = q_1^\pm \circ \Phi^\pm.$$

Since geometric control is invariant under $\Phi^\pm$, we can see that $q^\pm \neq 0$. By definition,

$$H_{p^\pm} q^\pm \big|_{(x,\xi)} = \frac{d}{ds} \left(q^\pm(x_s^\pm, \xi_s^\pm) \right) \big|_{s=0}.$$

Since $b^\pm$ is a constant of motion for the Hamiltonian system generated by $p^\pm$, it follows that

$$(\nabla_x b^\pm)(x_s^\pm, \xi_s^\pm) \dot{x}_s^\pm + (\nabla_\xi b^\pm)(x_s^\pm, \xi_s^\pm) \dot{\xi}_s^\pm = 0$$

for all s. Using this, we calculate that

$$
\begin{aligned}
\frac{d}{ds}\left(q^{\pm}(x_s^{\pm},\xi_s^{\pm})\right) &= \frac{d}{ds}\left(q_1^{\pm}\left(x_s^{\pm},\frac{\xi_s^{\pm}}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}\right)\right)\\
&= (\nabla_x q_1^{\pm})\left(x_s^{\pm},\frac{\xi_s^{\pm}}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}\right)\cdot(\dot{x}_s^{\pm})\\
+ (\nabla_\xi q_1^{\pm})\left(x_s^{\pm},\frac{\xi_s^{\pm}}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}\right) &\cdot \frac{(|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|\dot{\xi}_s^{\pm} - \xi_s^{\pm}\left((\nabla_x b^{\pm})(x_s^{\pm},\xi_s^{\pm})\dot{x}_s^{\pm} + (\nabla_\xi b^{\pm})(x_s^{\pm},\xi_s^{\pm})\dot{\xi}_s^{\pm}\right)}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|^2}\\
&= (\nabla_x q_1^{\pm})\left(x_s^{\pm},\frac{\xi_s^{\pm}}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}\right)\cdot(\nabla_\xi p^{\pm})(x_s^{\pm},\xi_s^{\pm})\\
&\quad - \frac{1}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}(\nabla_\xi q_1^{\pm})\left(x_s^{\pm},\frac{\xi_s^{\pm}}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}\right)\cdot(\nabla_x p^{\pm})(x_s^{\pm},\xi_s^{\pm})\\
&= (\nabla_x q_1^{\pm})\left(x_s^{\pm},\frac{\xi_s^{\pm}}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}\right)\cdot(\nabla_\xi p^{\pm})\left(x_s^{\pm},\frac{\xi_s^{\pm}}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}\right)\\
&\quad - (\nabla_\xi q_1^{\pm})\left(x_s^{\pm},\frac{\xi_s^{\pm}}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}\right)\cdot(\nabla_x p^{\pm})\left(x_s^{\pm},\frac{\xi_s^{\pm}}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}\right)\\
&= H_{p^{\pm}} q_1^{\pm}\Big|_{\left(x_s^{\pm},\frac{\xi_s^{\pm}}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}\right)},
\end{aligned}
$$

where we have used homogeneity to obtain that

$$
(\nabla_\xi p^{\pm})\left(x_s^{\pm},\xi_s^{\pm}\right) = (\nabla_\xi p^{\pm})\left(x_s^{\pm},\frac{\xi_s^{\pm}}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}\right)
$$

and

$$
\frac{1}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}(\nabla_x p^{\pm})\left(x_s^{\pm},\xi_s^{\pm}\right) = (\nabla_x p^{\pm})\left(x_s^{\pm},\frac{\xi_s^{\pm}}{|b^{\pm}(x_s^{\pm},\xi_s^{\pm})|}\right).
$$

If we define $V_R^{\pm} = (\Phi^{\pm})^{-1}\left(\dot{V}_R^{\pm}\right)$, then we have an open neighborhood of $\Omega_R^{\pm}$ such that

$$
H_{p^{\pm}} q^{\pm}\big|_{(x,\xi)} + C^{\pm}a(x) = H_{p^{\pm}} q_1^{\pm}\Big|_{\left(x,\frac{\xi}{|b^{\pm}(x,\xi)|}\right)} + C^{\pm}a(x) \gtrsim \left(\mathbb{1}_{\dot{V}_R^{\pm}}\circ\Phi^{\pm}\right)(x,\xi) \geq \mathbb{1}_{V_R^{\pm}},
$$

since $\Phi^{\pm}(V_R^{\pm}) \subset \dot{V}_R^{\pm}$. □

Now that we have completed step (1a), we move on to parts (1b) and (1c). Step (1b) pertains to non-trapped null bicharacteristics in the interior region. The symbol that we produce follows the construction appearing in [11] and utilized in many other works, such as [6] and [27]. Like in [27], we perform a factoring argument. The reason for studying the half-wave decomposition is due to the presence of a cutoff needed to make our constructed "symbol" genuinely symbolic. In the unfactored setting, cross terms in the metric arise when differentiating the cutoff in the computation of the Poisson bracket, generating an error term that is difficult to control. In the factored setting, this error can be handled straightforwardly.

Step (1c) takes place in the exterior region. This is of little concern, as we possess robust exterior estimates. We utilize this symbol as a means of bootstrapping the aforementioned error term, which will be compactly supported in the region where the exterior symbol has strictly positive Poisson bracket with $p^{\pm}$.

To these ends, we will analyze both half-waves simultaneously (as in Lemma 3.13). While this portion of the argument follows the one given in [27], it does require a modification; the escape function on interior, non-trapped null bicharacteristics needs an appropriate adjustment to ensure that it avoids trapped trajectories. We start with a proposition where we construct a function that will be used for the previously-described error absorption. The construction of this function comes from e.g. [27], [44].

Proposition 3.14. *Let $\sigma > 0$. Then, there exists $f \in C^\infty$ satisfying $f(r) \approx_\sigma 1$ when $r > R_0$ and $f'(r) \approx \sigma c_j 2^{-j} f(r)$ when $r \approx 2^j > R_0$.*

Here, (c_j) is the slow-varying sequence introduced in Section 1.2.

Remark 3.15. Although the sequence (c_j) is not defined for all natural numbers, the indices where it is not defined index finitely many dyadic regions (in particular, they omit where the operator P need not be a small AF perturbation). Since this region is compact, we can extend the sequence to such indices in an arbitrary manner. The typical way that this sequence is extended is by choosing c_j so that $\|g - m\|_{AF(A_j)} \lesssim c_j$ for the previously-undefined indices j. ■

Proof. As in [44], we can construct a smooth function $c(s)$ from the sequence (c_j) such that $c(s) \in (c_j, 2c_j)$ for each $s \in (2^j, 2^{j+1})$ and $|c'(s)| \leq \delta s^{-1} c(s)$. Since (c_j) is a positive sequence which converges to zero, it has a positive maximum, say c_N. Then, we observe that

$$\mathbf{c} \lesssim c_N \leq c(2^N + 2^{N-1}) = \left| \int_{2^N+2^{N-1}}^{\infty} c'(s)\, ds \right| \lesssim \int_1^\infty \frac{c(s)}{s}\, ds$$

and

$$\int_1^\infty \frac{c(s)}{s}\, ds \leq \sum_{j=0}^{\infty} \int_{2^j}^{2^{j+1}} \frac{2c_j}{2^j}\, ds = 2\sum_{j=0}^{\infty} c_j \lesssim \mathbf{c}.$$

That is,

$$\int_1^\infty \frac{c(s)}{s}\, ds \approx \mathbf{c}.$$

Now, set

$$f(r) = \exp\left(\sigma \int_1^r \frac{c(s)}{s}\, ds\right).$$

From our prior estimate, it is immediate that

$$f(r) \approx e^{\sigma \mathbf{c}} \approx_\sigma 1$$

for $r > R_0$, and

$$f'(r) = \sigma \frac{c(r)}{r} f(r) \approx \sigma c_j 2^{-j} f(r)$$

for $r \approx 2^j$. □

Now, we complete steps (1b) and (1c).

Lemma 3.16 (Non-trapped Escape Function Construction)**.** *Let $R \geq R_0$. Then, there exist $q^\pm \in C^\infty(T^*\mathbb{R}^3 \setminus o)$ and $W^\pm \subset \Omega_\infty^\pm$ so that $V_R^\pm \cup W^\pm = T^*\mathbb{R}^3 \setminus o$ and*

$$H_{p^\pm} q^\pm \gtrsim c_j 2^{-j} \mathbb{1}_{W^\pm}, \qquad |x| \approx 2^j.$$

Further, $q^\pm = \varepsilon q_{in}^\pm + q_{out}^\pm$, where $q_{in}^\pm = \tilde{q}_{in}^\pm \circ \Phi^\pm$ with $\tilde{q}_{in}^\pm \in C^\infty(T^*\mathbb{R}^3 \setminus o)$ is supported in $\{|x| \leq 4R\}$, $q_{out}^\pm \in S^0_{\text{hom}}(T^*\mathbb{R}^3 \setminus o)$, and $\varepsilon > 0$ is sufficiently small.

The inclusion of the sequence (c_j) is necessitated by the prior proposition, which is used for bootstrapping purposes in the exterior region. Its slowly varying nature allows one to work in the weight $\langle x\rangle^{-2}$ from the powers $|x| \approx 2^{-j}$ which will arise in the exterior (there is no trouble working in the weight $\langle x\rangle^{-2}$ in the interior region by compactness).

Proof. Choose $\psi^{\pm} \in C_c^{\infty}(T^*\mathbb{R}^3 \setminus o)$ such that

$$\operatorname{supp}\psi^{\pm} \subset \Omega_{\infty}^{\pm} \cap \{|x| \leq R\} \cap \Phi^{\pm}(T^*\mathbb{R}^3 \setminus o),$$
$$\psi^{\pm} \equiv 1 \text{ on } U_R^{\pm} := \big(\Omega_{\infty}^{\pm} \cap \{|x| \leq R\} \cap \Phi^{\pm}(T^*\mathbb{R}^3 \setminus o)\big) \setminus \dot{V}_R,$$

where $R \geq R_0$. Now, we define the function

$$\tilde{q}_{in}^{\pm}(x,\xi) = -\chi_{<2R}(|x|) \int_0^{\infty} \psi^{\pm} \circ \varphi_s^{\pm}(x,\xi)\, ds, \qquad (x,\xi) \in T^*\mathbb{R}^3 \setminus o.$$

Since non-trapped null bicharacteristic rays must exit any compact set after a finite amount of time, this integral is well-defined for each $(x,\xi) \in T^*\mathbb{R}^3 \setminus o$, which establishes $\tilde{q}_{in}^{\pm}$ as a well-defined function. It takes more work to show that $\tilde{q}_{in}^{\pm}$ is smooth. Similar to [6], we will begin by establishing a maximal amount of time that bicharacteristic rays can remain in the support of the integrand. We already know that $\operatorname{supp}\psi^{\pm}$ is compact. Let $V^{\pm}$ be an open neighborhood of $\operatorname{supp}\psi^{\pm}$ such that $\overline{V^{\pm}} \subset \Omega_{\infty}^{\pm}$. Take $\overline{V^{\pm}} = K$ in Proposition 3.11(c), and let T' be as given in the proposition.

We claim that every point $w^{\pm} \in T^*\mathbb{R}^3 \setminus o$ has a neighborhood $U_{w^{\pm}}$ of $w^{\pm}$ and a time $s_{w^{\pm}} \geq 0$ such that $(\psi^{\pm} \circ \varphi_s^{\pm})(z) = 0$ for every $z \in U_{w^{\pm}}$ and $s \in \mathbb{R}_+ \setminus [s_{w^{\pm}}, s_{w^{\pm}} + T']$. That is, all bicharacteristics (with speed ≈ 1) can spend no more than time T' within $\operatorname{supp}\psi^{\pm}$. The time $s_{w^{\pm}}$ bears no similarity to the variable of the same name in the proof of Proposition 3.13.

As a direct consequence of Proposition 3.11, we may take $U_{w^{\pm}} = V^{\pm}$ and $s_{w^{\pm}} = 0$ whenever $w^{\pm} \in \operatorname{supp}\psi^{\pm} \subset V^{\pm}$. If

$$w^{\pm} \notin \bigcup_{s \in \mathbb{R}} \varphi_{-s}^{\pm}\left(\operatorname{supp}\psi^{\pm}\right) =: \mathcal{X}^{\pm},$$

then the fact that $\mathcal{X}^{\pm}$ is closed provides an open neighborhood $U_{w^{\pm}}$ of $w^{\pm}$ such that $\mathcal{X}^{\pm} \cap U_{w^{\pm}} = \emptyset$. For each $z \in U_{w^{\pm}}$, we have that $\varphi_s^{\pm}(z) \notin \operatorname{supp}\psi^{\pm}$ for all $s \in \mathbb{R}$, i.e. $(\psi^{\pm} \circ \varphi_s^{\pm})(z) = 0$ for $s \in \mathbb{R}$. Hence, this case holds with $U_{w^{\pm}}$ as defined and $s_{w^{\pm}} = 0$. Finally, let $w^{\pm} \in \mathcal{X}^{\pm} \setminus \operatorname{supp}\psi^{\pm}$. Then, $\varphi_{s'}^{\pm}(w^{\pm}) \in \operatorname{supp}\psi^{\pm}$ for some $s' \in \mathbb{R} \setminus \{0\}$. If $s' > 0$, then we can combine this with the fact that $\varphi_0^{\pm}(w) \notin \operatorname{supp}\psi^{\pm}$ and the continuity of the flow to obtain $s_{w^{\pm}} > 0$ such that $\varphi_{s_{w^{\pm}}}^{\pm}(w) \in V^{\pm}$ and $\varphi_s^{\pm}(w) \notin \operatorname{supp}\psi^{\pm}$ for all $s \in [0, s_{w^{\pm}}]$. By continuity of the flow in the data, we can extend the above to a neighborhood $U_{w^\pm}$. That is, there exists a neighborhood $U_{w^\pm}$ of $w^\pm$ so that for all $z \in U_{w^\pm}$, we have that $\varphi^\pm_{s_{w^\pm}}(z) \in V^\pm$ and $(\psi^\pm \circ \varphi^\pm_s)(z) = 0$ for all $s \in [0, s_{w^\pm}]$. Applying Proposition 3.11 to $K = V^\pm$ implies that $(\psi^\pm \circ \varphi^\pm_s)(z) = 0$ for all $z \in U_{w^\pm}$ and $s \in [0, s_{w^\pm}] \cup [s_{w^\pm} + T', \infty)$. It remains to consider if we cannot assume that $s' > 0$. In this case,

$$w^\pm \notin \bigcup_{s \in \mathbb{R}_+} \varphi^\pm_{-s}(\operatorname{supp} \psi^\pm) =: \mathcal{X}^\pm_-$$

Note that $\mathcal{X}^\pm_-$ is closed by the same logic which showed that $\mathcal{X}^\pm$ is closed (see the proof in Proposition 3.11(c)). From here, one can simply proceed as in the case where $w^\pm \notin \mathcal{X}^\pm$.

Using this result, we know that the integral present in $\tilde{q}^\pm_{in}$ is always over an interval of maximal length T'. Hence, differentiation under the integral sign is non-problematic and in view of the regularity of the flow map, we conclude that $\tilde{q}^\pm_{in} \in C^\infty(T^*\mathbb{R}^3 \setminus o)$. Additionally, it is supported in $\{|x| \leq 4R\}$. In particular, it is smooth and bounded in all derivatives on the compact set

$$\{|x| \leq 4R\} \cap \Phi^\pm(T^*\mathbb{R}^3 \setminus o).$$

Now, consider the smooth function

$$q^\pm_{in} = \tilde{q}^\pm_{in} \circ \Phi^\pm$$

defined on $T^*\mathbb{R}^3 \setminus o$. As in the proof of Lemma 3.13, we get that

$$H_{p^\pm} q^\pm_{in}\big|_{(x,\xi)} = H_{p^\pm} \tilde{q}^\pm_{in}\big|_{\Phi^\pm(x,\xi)}.$$

Now, we calculate that

$$\begin{aligned} H_{p^\pm} \tilde{q}^\pm_{in}\big|_{\Phi^\pm(x,\xi)} &= \chi_{<2R}(|x|)\psi^\pm\left(x, \frac{\xi}{|b^\pm(x,\xi)|}\right) \\ &\quad + \frac{1}{2R} b^\pm_{\xi_k}\left(x, \frac{\xi}{|b^\pm(x,\xi)|}\right) \frac{x_k}{|x|} \chi'\left(\frac{|x|}{2R}\right) \int_0^\infty \psi^\pm \circ \varphi^\pm_s\left(x, \frac{\xi}{|b^\pm(x,\xi)|}\right) ds. \end{aligned}$$

The first term is non-negative, supported in $\Omega^\pm_\infty \cap \{|x| \leq R\}$, and equal to 1 on $U^\pm := \Phi^{-1}(\dot{U}^\pm_R)$.

The second term is an error term which is supported in $\{2R \leq |x| \leq 4R\}$. The primary purpose of the exterior multiplier is to absorb this error term. To that end, let

$$q_{out}^{\pm} = -\chi_{>R}(|x|) f(|x|) b_{\xi_k}^{\pm} \frac{x_k}{|x|},$$

where f is the function constructed in Proposition 3.14. It is easy to see that $q_{out}^{\pm} \in S^0_{\mathrm{hom}}(T^*\mathbb{R}^3 \setminus o)$, as it is smooth, bounded in all x derivatives due to asymptotic flatness, homogeneous of degree 0, and satisfies the appropriate symbol estimate. One can readily compute that

$$\begin{aligned} H_{p^{\pm}} q_{out}^{\pm} = {} & b_{\xi_k}^{\pm} \frac{x_k}{|x|} \chi_{>R}(|x|) f'(|x|) b_{\xi_j}^{\pm} \frac{x_j}{|x|} + b_{\xi_k}^{\pm} \left(\delta_{jk} - \frac{x_j x_k}{|x|^2} \right) \chi_{>R}(|x|) \frac{f(|x|)}{|x|} \left(\delta_{jl} - \frac{x_j x_l}{|x|^2} \right) b_{\xi_l}^{\pm} \\ & + R^{-1} \chi' \left(\frac{|x|}{R} \right) b_{\xi_k}^{\pm} \frac{x_k}{|x|} f(|x|) b_{\xi_j}^{\pm} \frac{x_j}{|x|} + \mathcal{O}(\langle x \rangle |\partial g|) \chi_{>R}(|x|) |x|^{-1}. \end{aligned}$$

We remark that the last term is small for $|x| > R$ by asymptotic flatness (and it is localized to this region due to the cutoff), while the remaining terms are all non-negative. The third term is non-negative and supported in the annulus $\{R \leq |x| \leq 2R\}$ due to the support of χ'. Making σ large enough and using asymptotic flatness provides that, for any $|x| \approx 2^j$,

$$\begin{aligned} H_{p^{\pm}} q_{out}^{\pm} & > \frac{\sigma}{2} c_j 2^{-j} f(|x|) \chi_{>R}(|x|) \frac{|x \cdot \nabla_\xi b^{\pm}|^2}{|x|^2} + \chi_{>R}(|x|) \frac{f(|x|)}{|x|} \left(|\nabla_\xi b^{\pm}|^2 - \frac{|x \cdot \nabla_\xi b^{\pm}|^2}{|x|^2} \right) \\ & \gtrsim c_j 2^{-j} \chi_{>R}(|x|) |\nabla_\xi b^{\pm}|^2 \\ & \gtrsim c_j 2^{-j} \chi_{>R}(|x|). \end{aligned}$$

Thus, $H_{p^{\pm}} q_{out}^{\pm}$ is non-negative, strictly positive for $|x| > R$, and

$$H_{p^{\pm}} q_{out}^{\pm} \gtrsim c_j 2^{-j} \chi_{>R}, \qquad |x| \approx 2^j.$$

Recall that the error term in $H_{p^{\pm}} q_{in}^{\pm}$ is bounded and supported in $\{2R \leq |x| \leq 4R\}$, and $H_{p^{\pm}} q_{out}^{\pm}$ is strictly positive on the support of this error (with a uniform bound from below on this set).

Define

$$q^{\pm} = \varepsilon q_{in}^{\pm} + q_{out}^{\pm} \in C^\infty(T^*\mathbb{R}^3 \setminus o),$$

where $0 < \varepsilon \ll 1$. By choosing ε sufficiently small, we may absorb the aforementioned error due to our prior discussion, obtaining that $H_{p^\pm} q^\pm$ is non-negative everywhere and positive on

$$W^\pm := U^\pm \cup \{(x,\xi) \in T^*\mathbb{R}^3 \setminus o : |x| > R\}.$$

By Proposition 3.11(a),

$$\begin{aligned} V_R^\pm \cup U^\pm &= V_R^\pm \cup \left((\Omega_\infty^\pm \cap \{|x| \leq R\}) \setminus V_R^\pm\right) \\ &\supset \left(\Omega_R^\pm \cup \Omega_\infty\right) \cap \{|x| \leq R\} \\ &= \left(T^*\mathbb{R}^3 \setminus o\right) \cap \{|x| \leq R\}, \end{aligned}$$

and so

$$\begin{aligned} V_R^\pm \cup U^\pm &= \left(T^*\mathbb{R}^3 \setminus o\right) \cap \{|x| \leq R\}, \\ V_R^\pm \cup W^\pm &= T^*\mathbb{R}^3 \setminus o. \end{aligned}$$

We have already shown that

$$H_{p^\pm} q^\pm \approx 1 \qquad (x,\xi) \in U^\pm$$

and

$$H_{p^\pm} q^\pm \gtrsim c_j 2^{-j} \chi_{>R}, \qquad |x| \approx 2^j.$$

The latter estimate readily extends to

$$H_{p^\pm} q^\pm \gtrsim c_j 2^{-j} \mathbb{1}_{W^\pm}, \qquad |x| \approx 2^j$$

by the compactness of the interior region $\{|x| \leq R\}$. □

Now, we combine on the light cones to get our desired symbol q, as well as obtain positivity on the elliptic set (step (2)). This largely follows the steps present in [27], although we have additional technicalities resulting from the damping.

(Proof of Lemma 3.4). Let $q_1^\pm$ denote the symbol $q^\pm$ constructed in Lemma 3.13 (not the symbol $q_1^\pm$ from the same lemma) and $q_2^\pm$ denote the symbol $q^\pm$ constructed in Lemma 3.16. We remark

that, as a consequence of the chain rule, both symbols satisfy the standard S^0 bounds for $|\xi| \geq 1$. First, we truncate to the high-frequency regime via the symbols

$$q^{\pm}_{j,>\lambda} = e^{\sigma q^{\pm}_j} \chi_{>\lambda}(|b^{\pm}|),\ j = 1, 2,$$

where σ is the parameter in Proposition 3.14. We assume that $\lambda > 1$. The exponentiation is implemented for bootstrapping: taking derivatives of the exponential will provide multiplication by $\sigma \gg 1$. Since $|b^{\pm}| \approx |\xi|$, these cutoffs genuinely truncate to high frequencies when λ is large. Further, the truncation to $|\xi| \gtrsim 1$ eliminates the singularities of $q^{\pm}_j$, i.e. $q^{\pm}_j \chi_{>\lambda}(|b^{\pm}|)$ smoothly extends to an element of $S^0(T^*\mathbb{R}^3)$.

We claim that exponentiation preserves the symbol class, so that $q^{\pm}_{j,>\lambda} \in S^0(T^*\mathbb{R}^3)$. We can immediately see that $q^{\pm}_{j,>\lambda}$ is smooth. Note that for $|\xi| \geq \lambda$, the exponentials $e^{\sigma q^{\pm}_j}$ are bounded since $q^{\pm}_j$ are bounded, and for $|\xi| < \lambda$, we immediately have that $q^{\pm}_{j,>\lambda} \equiv 0$. When checking the symbolic nature of $q^{\pm}_{j,>\lambda}$, we only need to study the boundedness of the ξ derivatives since our symbols $q^{\pm}_j$ are bounded in all derivatives in x. Taking a partial derivative in ξ provides that

$$\partial_{\xi_k} q^{\pm}_{j,>\lambda} = \sigma e^{\sigma q^{\pm}_j} (\partial_{\xi_k} q^{\pm}_j) \chi_{>\lambda}(|b^{\pm}|) \mp \frac{e^{\sigma q^{\pm}_j}}{\lambda} (\partial_{\xi_k} b^{\pm}) \chi' \left(\frac{|b^{\pm}|}{\lambda} \right).$$

The first term is $\mathcal{O}(\langle \xi \rangle^{-1})$, and the second term is compactly supported in ξ. Due to the aforementioned compact support, we only need consider further ξ differentiation of $\sigma e^{\sigma q^{\pm}_j} (\partial_{\xi_k} q^{\pm}_j) \chi_{>\lambda}(|b^{\pm}|)$. If the ξ derivative lands on the exponential, then the result is $\mathcal{O}(\langle \xi \rangle^{-2})$ by the prior argument. If the derivative lands on $\partial_\xi q^{\pm}_j$, then the same asymptotics hold since $\partial_\xi q^{\pm}_j \in S^{-1}(T^*\mathbb{R}^3)$. If the derivative lands on the cutoff, then the result is compactly-supported in ξ . This establishes that $q^{\pm}_{j,>\lambda} \in S^0(T^*\mathbb{R}^3)$.

Now, we combine the symbols constructed on each light cones together as

$$q(\tau, x, \xi) = (\tau - b^+)(q^-_{1,>\lambda} + q^-_{2,>\lambda}) + (\tau - b^-)(q^+_{1,>\lambda} + q^+_{2,>\lambda}).$$

Calling

$$q_j = (\tau - b^+) q^-_{j,>\lambda} + (\tau - b^-) q^+_{j,>\lambda},$$

we can see that

$$\begin{aligned}(H_p q + 2\gamma\tau a q)\big|_{\tau=b^\pm} &= (H_p q_1 + 2\gamma\tau a q_1)\big|_{\tau=b^\pm} + (H_p q_2 + 2\gamma\tau a q_2)\big|_{\tau=b^\pm} \\ &= H_p q_1\big|_{\tau=b^\pm} \pm 2\gamma b^\pm (b^+ - b^-) a q^\pm_{1,>\lambda} \\ &\qquad + H_p q_2\big|_{\tau=b^\pm} \pm 2\gamma b^\pm (b^+ - b^-) a q^\pm_{2,>\lambda}.\end{aligned}$$

We will work with each term in the last equality separately. First, we compute that

$$\begin{aligned}H_p q_j\big|_{\tau=b^\pm} &= (b^+ - b^-)^2 H_{p^\pm} q^\pm_{j,>\lambda} \\ &\qquad + (b^\pm - b^\mp) q^\pm_{j,>\lambda} (b^\pm_{\xi_j} b^\mp_{x_j} - b^\pm_{x_j} b^\mp_{\xi_j}) \\ &= \sigma (b^+ - b^-)^2 q^\pm_{j,>\lambda} H_{p^\pm} q^\pm_j \\ &\qquad + (b^\pm - b^\mp) q^\pm_{j,>\lambda} (b^\pm_{\xi_j} b^\mp_{x_j} - b^\pm_{x_j} b^\mp_{\xi_j}).\end{aligned}$$

By making σ sufficiently large, we get that

$$H_p q_j\big|_{\tau=b^\pm} \geq \frac{1}{2}\sigma (b^+ - b^-)^2 q^\pm_{j,>\lambda} H_{p^\pm} q^\pm_j + E^\pm_j,$$

where $E^\pm_j$ are error terms which are supported in a neighborhood of the region where $H_{p^\pm} q^\pm_j = 0$. These terms are non-problematic, as they are readily absorbed into the above estimate with differing j when we combine the estimates together. Hence, we will drop the $E^\pm_j$'s for ease of notation.

Observe that

$$\frac{b^\pm}{b^\pm - b^\mp} \approx 1.$$

By choosing γ large enough, we may apply Lemma 3.13 to obtain that

$$\begin{aligned}(H_p q_1 + 2\gamma\tau a q_1)\big|_{\tau=b^\pm} &\geq \frac{1}{2}\sigma (b^+ - b^-)^2 q^\pm_{1,>\lambda} H_{p^\pm} q^\pm_1 \pm 2\gamma b^\pm (b^+ - b^-) a q^\pm_{1,>\lambda} \\ &= \frac{1}{2}\sigma (b^+ - b^-)^2 q^\pm_{1,>\lambda} \left(H_{p^\pm} q^\pm_1 + \left(\frac{4\gamma}{\sigma}\right) \frac{b^\pm}{b^\pm - b^\mp} a \right) \\ &\gtrsim |\xi|^2 q^\pm_{1,>\lambda} \left(H_{p^\pm} q^\pm_1 + \frac{\gamma}{\sigma} a \right) \\ &\gtrsim \mathbb{1}_{|\xi|\geq\lambda} \mathbb{1}_{V^\pm_R} |\xi|^2.\end{aligned} \tag{3.6}$$

Notice that γ depends on $\mathbf{c}$ (and σ).

For the $j = 2$ term, we use the prior computation, the fact that the damping term has positive sign, and Lemma 3.16:

$$(3.7)\qquad (H_p q_2 + 2\gamma\tau a q_2)\big|_{\tau=b^\pm} \geq \frac{1}{2}\sigma(b^+ - b^-)^2 q_{2,>\lambda}^\pm H_{p^\pm} q_2^\pm \pm 2\gamma b^\pm (b^+ - b^-) a q_{2,>\lambda}^\pm$$
$$\gtrsim \mathbb{1}_{|\xi|\geq\lambda} \mathbb{1}_{W^\pm} c_j 2^{-j} |\xi|^2, \qquad |x| \approx 2^j.$$

Recall that $V_R^\pm \cup W^\pm = T^*\mathbb{R}^3 \setminus o$. Combining (3.6) and (3.7) together, we conclude that

$$(H_p q + 2\gamma\tau a q)\big|_{\tau=b^\pm} \gtrsim \mathbb{1}_{|\xi|\geq\lambda} \langle x\rangle^{-2} |\xi|^2,$$

where we have used the slowly-varying, summable nature of (c_j).

This provides the desired bound over the characteristic set. To extend it to all of phase space, we must construct a lower-order correction term. Explicitly, we seek an $m \in S^0$ so that

$$H_p q + 2\gamma\tau a q + mp \gtrsim \mathbb{1}_{|\xi|\geq\lambda} \langle x\rangle^{-2} |\xi|^2.$$

If we write

$$H_p q + 2\gamma a \tau q = a_0 \tau^2 + a_1 \tau + a_2,$$

where $a_j \in S^j$, then we have already established that

$$(3.8)\qquad a_0(x,\xi)(b^\pm(x,\xi))^2 + a_1(x,\xi) b^\pm(x,\xi) + a_2(x,\xi) \gtrsim \mathbb{1}_{|\xi|\geq\lambda} \langle x\rangle^{-2} |\xi|^2$$

So, we must analyze the quantity

$$a_0\tau^2 + a_1\tau + a_2 + pm = (a_0 + m)\tau^2 + (a_1 - (b^+ + b^-)m)\tau + (a_2 + b^+ b^- m).$$

If we choose m so that

$$(3.9)\qquad a_0 + m > 0, \qquad |\xi| \geq \lambda$$

and

$$(a_1 - (b^+ + b^-)m)^2 - 4(a_0 + m)(a_2 + b^+ b^- m) < 0, \qquad |\xi| \geq \lambda, \tag{3.10}$$

then we will have that $a_0\tau^2 + a_1\tau + a_2 + mp$ is positive for $|\xi| \geq \lambda$ (the first condition on m guarantees that this polynomial in τ is concave up, and the second guarantees that there are no real zeros).

Let us begin by focusing on (3.10). The function

$$\begin{aligned} P(m) &= (a_1 - (b^+ + b^-)m)^2 - 4(a_0 + m)(a_2 + b^+ b^- m) \\ &= (b^+ - b^-)^2 m^2 - (2a_1(b^+ + b^-) + 4a_0 b^+ b^- + 4a_2)m + (a_1^2 - 4a_0 a_2) \end{aligned}$$

is a quadratic polynomial in m with a positive coefficient on the quadratic term, so it will achieve a minimal value at

$$m = \frac{a_1(b^+ + b^-) + 2(a_0 b^+ b^- + a_2)}{(b^+ - b^-)^2}.$$

It is readily seen that $m \in S^0$ and that m is supported where $|\xi| \geq \lambda$. This minimal value is

$$\begin{aligned} P(m) = &\left(a_1 - (b^+ - b^-)\frac{a_1(b^+ + b^-) + 2(a_0 b^+ b^- + a_2)}{(b^+ - b^-)^2}\right)^2 \\ &- 4\left(a_0 + \frac{a_1(b^+ + b^-) + 2(a_0 b^+ b^- + a_2)}{(b^+ - b^-)^2}\right)\left(a_2 + b^+ b^- \frac{a_1(b^+ + b^-) + 2(a_0 b^+ b^- + a_2)}{(b^+ - b^-)^2}\right) \\ = &-4\frac{(a_0(b^+)^2 + a_1 b^+ + a_2)(a_0(b^-)^2 + a_1 b^- + a_2)}{(b^+ - b^-)^2} \\ = &-4(b^+ - b^-)^{-2}\left((H_p q + 2\gamma\tau a q)\big|_{\tau=b^+}\right)\left(H_p q + 2\gamma\tau a q)\big|_{\tau=b^-}\right) \\ < &\ 0, \end{aligned}$$

where we have used (3.8). So, (3.10) is satisfied. To establish (3.9), one can readily check that

$$\begin{aligned} a_0 + m &= a_0 + \frac{a_1(b^+ + b^-) + 2(a_0 b^+ b^- + a_2)}{(b^+ - b^-)^2} \\ &= (b^+ - b^-)^{-2}\left((H_p q + 2\gamma\tau a q)\big|_{\tau=b^+} + (H_p q + 2\gamma\tau a q)\big|_{\tau=b^-}\right) \\ &> 0 \end{aligned}$$

for $|\xi| \geq \lambda$.

This gives us that

$$H_p q + 2\gamma\tau a q + mp > 0$$

for $|\xi| \geq \lambda$. In fact, we can check that the minimal value of $H_p q + 2\gamma\tau a q + mp$ in τ for $|\xi| \geq \lambda$ is

$$\frac{(a_0(b^+)^2 + a_1 b^+ + a_2)(a_0(b^+)^2 + a_1 b^+ + a_2)}{(a_0(b^+)^2 + a_1 b^+ + a_2) + (a_0(b^-)^2 + a_1 b^- + a_2)} = \frac{\left((H_p q + 2\gamma\tau a q)\big|_{\tau=b^+}\right)\left(H_p q + 2\gamma\tau a q)\big|_{\tau=b^-}\right)}{(H_p q + 2\gamma\tau a q)\big|_{\tau=b^+} + (H_p q + 2\gamma\tau a q)\big|_{\tau=b^-}}.$$

The numerator is bounded below by $\langle x\rangle^{-4} |\xi|^4$. In view of the support and symbolic properties of q, the denominator satisfies the bounds

$$(H_p q + 2\gamma\tau a q)\big|_{\tau=b^+} + (H_p q + 2\gamma\tau a q)\big|_{\tau=b^-} \approx \langle x\rangle^{-2} |\xi|^2.$$

Since $|b^{\pm}(x,\xi)| \approx |\xi|$ and $|\tau| = |b^{\pm}(x,\xi)|$ in the above, we conclude the desired result.

$\square$

3.5 Case Reductions

In this section, we will reduce the proof of Theorem 1.6 to a simpler problem.

Case Reduction #1: It is sufficient to prove Theorem 1.6 when the initial data and forcing are supported in $\{|x| \leq 2R_0\}$.

Proof. Call the forcing term f. Let $\tilde{P}$ be a small AF perturbation of $\Box$ which agrees with P for $|x| > R_0$, and suppose that v solves

$$\tilde{P}v = f, \qquad v[0] = u[0].$$

Consider the function $\tilde{u} = u - \chi_{>R_0} v$ which has the following properties:

(a) Its Cauchy data is compactly supported in $\{|x| \leq 2R_0\}$: Indeed, $\tilde{u}[0] = u[0] - \chi_{>R_0} v[0]$, which is $u[0]$ for $|x| \leq R_0$ and zero for $|x| > 2R_0$.

(b) $P\tilde{u}$ is compactly supported in $\{|x| \leq 2R_0\}$: Indeed,

$$P\tilde{u} = f - P(\chi_{>R_0} v),$$

which is f for $|x| \le R_0$ and zero for $|x| > 2R_0$. More specifically,

$$\begin{aligned} P\tilde{u} &= f - \chi_{>R_0} P v + \mathcal{O}(R_0^{-1})\chi'\left(\frac{|x|}{R_0}\right)|\nabla v| + \mathcal{O}(R_0^{-2})\chi''\left(\frac{|x|}{R_0}\right) v \\ &= f - \chi_{>R_0} f + \mathcal{O}(R_0^{-1})\chi'\left(\frac{|x|}{R_0}\right)|\nabla v| + \mathcal{O}(R_0^{-2})\chi''\left(\frac{|x|}{R_0}\right) v. \end{aligned}$$

We also record that

$$[\tilde{P}, \chi_{>R_0}]v = \mathcal{O}(R_0^{-1})\chi'\left(\frac{|x|}{R_0}\right)|\nabla v| + \mathcal{O}(R_0^{-2})\chi''\left(\frac{|x|}{R_0}\right) v.$$

Since $\tilde{P}$ is a small AF perturbation of $\Box$, local energy decay holds for $\tilde{P}$. Applying the local energy decay estimate to functions $\chi_{>R_0} v$ and v gives

$$\|\chi_{>R_0} v\|_{LE^1[0,T]} + \|\partial(\chi_{>R_0} v)\|_{L_t^\infty L_x^2[0,T]} \lesssim \|\partial(\chi_{>R_0} v)(0)\|_{L^2} + \|f\|_{LE^* + L_t^1 L_x^2[0,T]} + \|v\|_{LE^1_{R_0}[0,T]},$$

and

$$\|v\|_{LE^1[0,T]} + \|\partial v\|_{L_t^\infty L_x^2[0,T]} \lesssim \|\partial u(0)\|_{L^2} + \|f\|_{LE^* + L_t^1 L_x^2[0,T]},$$

respectively. Notice that the prior commutator estimate was used in the first of the two inequalities above. Combining these two estimates and utilizing the Hardy inequality on the term $\|(\nabla \chi_{>R_0}) v(0)\|_{L^2}$ yields that

$$\|\chi_{>R_0} v\|_{LE^1[0,T]} + \|\partial(\chi_{>R_0}) v\|_{L_t^\infty L_x^2[0,T]} \lesssim \|\partial u(0)\|_{L^2} + \|f\|_{LE^* + L_t^1 L_x^2[0,T]}.$$

We claim that it suffices to prove (1.4) for $\tilde{u}$. If such an estimate held for $\tilde{u}$, then

$$\begin{aligned} &\|u\|_{LE^1[0,T]} + \|\partial u\|_{L_t^\infty L_x^2[0,T]} \le \|\tilde{u}\|_{LE^1[0,T]} + \|\partial \tilde{u}\|_{L_t^\infty L_x^2[0,T]} + \|\chi_{>R_0} v\|_{LE^1[0,T]} + \|\partial(\chi_{>R_0}) v\|_{L_t^\infty L_x^2[0,T]} \\ &\lesssim \|\partial \tilde{u}(0)\|_{L^2} + \left\|\langle x\rangle^{-2} \tilde{u}\right\|_{LE[0,T]} + \|P\tilde{u}\|_{LE^* + L_t^1 L_x^2[0,T]} + \|\partial u(0)\|_{L^2} + \|f\|_{LE^* + L_t^1 L_x^2[0,T]} \\ &\lesssim \|\partial u(0)\|_{L^2} + \left\|\langle x\rangle^{-2} \tilde{u}\right\|_{LE[0,T]} + \|f\|_{LE^* + L_t^1 L_x^2[0,T]}. \end{aligned}$$

The only term which we have not shown how to deal with is the middle term on the right-hand side of the prior inequality, which is relatively straightforward:

$$\left\|\langle x\rangle^{-2}\tilde{u}\right\|_{LE[0,T]} \le \left\|\langle x\rangle^{-2}u\right\|_{LE[0,T]} + \left\|\langle x\rangle^{-2}\chi_{>R_0}v\right\|_{LE} \lesssim \left\|\langle x\rangle^{-2}u\right\|_{LE[0,T]} + R_0^{-1}\left\|\chi_{>R_0}v\right\|_{LE^1[0,T]}$$
$$\lesssim \|\partial u(0)\|_{L^2} + \left\|\langle x\rangle^{-2}u\right\|_{LE[0,T]} + \|f\|_{LE^*+L_t^1L_x^2[0,T]}.$$

Thus, it suffices to prove (1.4) for $\tilde{u}$. □

Case Reduction #2: It suffices to prove Theorem 1.6 when $u[0]=0$ and $f\in LE_c^*$.

Proof. Suppose that we have the estimate

$$\|u\|_{LE^1[0,T]} + \|\partial u\|_{L_t^\infty L_x^2[0,T]} \lesssim \left\|\langle x\rangle^{-2}u\right\|_{LE} + \|Pu\|_{LE_c^*[0,T]}, \tag{3.11}$$

with $u[0]=0$. Set up the equation $Pw=g\in L_t^1L_x^2$, with $w[0]$ non-trivial. We can write $w=\sum_k w_k$, where w_k solves

$$Pw_k = \mathbb{1}_{[k,k+1]}(t)g,$$

with $w_0[0]=w[0]$ and $w_k[0]=0$ for $k>0$. By uniqueness, if $k>0$, then w_k vanishes for times $t<k$. We will approximate w with $\sum_k \beta_k(t)w_k$, where $\beta_k\in C_c^\infty([0,\infty))$ and $\beta_k\equiv 1$ on $[k,k+1]$ and identically zero for $t\le k-1$ (except for $k=0$) and $t\ge k+2$.

We begin by establishing the estimate

$$\sum_k \left(\|\beta_k w_k\|_{LE^1[0,T]} + \|\partial(\beta_k w_k)\|_{L_t^\infty L_x^2[0,T]}\right) \lesssim \|\partial w(0)\|_{L^2} + \sum_k \int_k^{k+1} \|g(s)\|_{L^2}\, ds. \tag{3.12}$$

For the first term on the left,

$$\|\beta_k w_k\|_{LE^1[0,T]} \lesssim \|\partial(\beta_k w_k)\|_{LE[0,T]} + \left\|r^{-1}\beta_k w_k\right\|_{LE[0,T]}$$
$$\lesssim \|\partial(\beta_k w_k)\|_{L_t^\infty L_x^2[0,T]}.$$

To obtain the last inequality above, we bounded the first term by taking the supremum in t (note that the interval has length one) and the second term by using the Hardy inequality. Now, we work to bound $\|\partial(\beta_k w_k)\|_{L_t^\infty L_x^2}$. Note that

$$\|\partial(\beta_k w_k)\|_{L_t^\infty L_x^2[0,T]} \leq \|w_k \partial_t \beta_k\|_{L_t^\infty L_x^2[0,T]} + \|\beta_k \partial w_k\|_{L_t^\infty L_x^2[0,T]}.$$

We will estimate the first term on the right using the Fundamental Theorem of Calculus and the Minkowski integral inequality. For $k \geq 1$,

$$\begin{aligned}
\|w_k \partial_t \beta_k\|_{L_t^\infty L_x^2[0,T]} &= \sup_{t\in[0,T]} |\partial_t \beta_k| \left\| \int_{k-1}^{t} \partial_s w_k(s)\, ds \right\|_{L_x^2} \\
&\leq \sup_{t\in[0,T]} |\partial_t \beta_k| \int_{k-1}^{t} \|\partial_s w_k(s)\|_{L_x^2}\, ds \\
&\lesssim \|\partial_t w_k\|_{L_t^\infty L_x^2([k-1,k+2])},
\end{aligned}$$

and for $k = 0$,

$$\begin{aligned}
\|w_0 \partial_t \beta_k\|_{L_t^\infty L_x^2[0,T]} &= \sup_{t\in[0,T]} |\partial_t \beta_0| \left\| \int_{t}^{2} \partial_s w_0(s)\, ds \right\|_{L_x^2} \\
&\leq \sup_{t\in[0,T]} |\partial_t \beta_0| \int_{0}^{2} \|\partial_s w_0(s)\|_{L_x^2}\, ds \\
&\lesssim \|\partial_t w_0\|_{L_t^\infty L_x^2([0,2])},
\end{aligned}$$

both of which are bounded above by a multiple of $\|\partial w_k\|_{L_t^\infty L_x^2[0,k+2]}$. Via Corollary 2.3, we obtain that

$$\|\partial w_k\|_{L_t^\infty L_x^2[0,k+2]} \lesssim \|\partial w_k(0)\|_{L^2} + \|P w_k\|_{L_t^1 L_x^2[0,k+2]} \lesssim \int_{k}^{k+1} \|g(s)\|_{L^2}\, ds$$

when $k > 0$ and

$$\|\partial w_0\|_{L_t^\infty L_x^2[0,2]} \lesssim \|\partial w_0(0)\|_{L^2} + \|P w_0\|_{L_t^1 L_x^2[0,k+2]} \lesssim \|\partial w(0)\|_{L^2} + \int_{0}^{1} \|g(s)\|_{L^2}\, ds$$

when $k = 0$. Together, these provide the right-hand side of (3.12).

We now make another claim, namely the inequality

$$\left\| \langle x \rangle^{-2} \sum_k \beta_k w_k \right\|_{LE[0,T]} + \left\| P\left(w - \sum_k \beta_k w_k \right) \right\|_{LE^*[0,T]} \lesssim \sum_k \|\partial w_k\|_{L^\infty_t L^2_x([0,k+2])}. \tag{3.13}$$

Notice that the right-hand side can be subsequently attacked as performed previously. The first term on the left-hand side of (3.13) is bounded by the term on the right via the Hardy inequality (and taking a supremum in time). For the other term, we can write

$$P\left(w - \sum_k \beta_k w_k \right) = \sum_k \mathcal{O}(\beta_k')[w_k + \partial w_k] + \mathcal{O}(\beta_k'')w_k.$$

Since g is compactly-supported in space, and, for each k, β_k is supported on the unit scale in time, finite speed of propagation applied to w_k and ∂w_k guarantees compact spatial support of the above term independently of T (each is supported on a time scale of $\mathcal{O}(1)$). This allows us to remove the weight and the infinite sum in the LE^* norm and transition to an LE norm. Proceeding as with the first term on the left-hand side of (3.13) proves the claim.

Now, we can put our estimates fairly easily. Using (3.12), (3.13), and (3.11) applied to $w - \sum_k \beta_k w_k$, we finally obtain that

$$\begin{aligned}
\|w\|_{LE^1[0,T]} + \|\partial w\|_{L^\infty_t L^2_x[0,T]} &\leq \left\| \sum_k \beta_k w_k \right\|_{LE^1[0,T]} + \left\| w - \sum_k \beta_k w_k \right\|_{LE^1[0,T]} \\
&\quad + \left\| \partial \left(\sum_k \beta_k w_k \right) \right\|_{L^\infty_t L^2_x[0,T]} + \left\| \partial \left(w - \sum_k \beta_k w_k \right) \right\|_{L^\infty_t L^2_x[0,T]} \\
&\lesssim \|\partial w(0)\|_{L^2} + \sum_k \int_k^{k+1} \|g(s)\|_{L^2_x}\, ds \\
&\quad + \left\| \langle x \rangle^{-2} \left(w - \sum_k \beta_k w_k \right) \right\|_{LE[0,T]} \\
&\quad + \left\| P\left(w - \sum_k \beta_k w_k \right) \right\|_{LE^*_c[0,T]} \\
&\lesssim \|\partial w(0)\|_{L^2} + \left\| \langle x \rangle^{-2} w \right\|_{LE[0,T]} + \sum_k \int_k^{k+1} \|g(s)\|_{L^2_x}\, ds \\
&= \|\partial w(0)\|_{L^2} + \left\| \langle x \rangle^{-2} w \right\|_{LE} + \|g\|_{L^1_t L^2_x}.
\end{aligned}$$

□

In view of this case reduction and Corollary 2.3, it is enough to establish

$$\|u\|_{LE^1[0,T]} \lesssim \left\| \langle x \rangle^{-2} u \right\|_{LE[0,T]} + \|Pu\|_{LE_c^*[0,T]}, \tag{3.14}$$

in order to prove Theorem 1.6.

Case Reduction #3: It suffices to prove Theorem 1.6 for solutions who also have vanishing Cauchy data at $t = T$.

Proof. This follows from the same argument as above, except that we make $w_k[0] = 0$ for all k *except* the last k in our partition of unity, which is given Cauchy data of $u[T]$ at time T. This is readily handled via our coercive energy (in particular, we use Corollary 2.3). The implicit constant will have no dependence on T for this reason (as well as the unit intervals utilized in the partition of unity). □

Case Reduction #4: It suffices to prove Theorem 1.6 for u supported in $\{|x| \leq 2R_0\}$.

Proof. Write $u = \chi_{<R_0} u + \chi_{>R_0} u$. On the exterior piece $\chi_{>R_0} u$, we apply Proposition 2.5 and Corollary 2.3 to get that

$$\|\chi_{>R_0} u\|_{LE^1[0,T]} \lesssim R_0^{-1} \|\chi_{>R_0} u\|_{LE_{R_0}[0,T]} + \|P(\chi_{>R_0} u)\|_{LE^*_{>R_0}[0,T]}.$$

The first term on the right is directly bounded by $\|u\|_{LE^1_{R_0}[0,T]}$. For the second term, we write

$$P(\chi_{>R_0} u) = (\chi_{>R_0}) Pu + [P, \chi_{>R_0}] u,$$

and one can calculate that

$$[P, \chi_{>R_0}] u(t,x) = \mathcal{O}(R_0^{-1}) \chi' \left(\frac{|x|}{R_0} \right) \partial u(t,x) \mathcal{O}(R_0^{-2}) \chi'' \left(\frac{|x|}{R_0} \right) u(t,x).$$

In LE^*, this term bounded by $\|u\|_{LE^1_{R_0 \leq |\cdot| \leq 2R_0}[0,T]}$, and so we have

$$\|\chi_{>R_0} u\|_{LE^1[0,T]} \lesssim \|u\|_{LE^1_{R_0}[0,T]}.$$

Suppose that (3.14) holds for $\chi_{<R_0} u$. From this, we get the estimate

$$\|\chi_{<R_0} u\|_{LE^1[0,T]} \lesssim \left\| \langle x \rangle^{-2} \chi_{<R_0} u \right\|_{LE[0,T]} + \|Pu\|_{LE_c^*[0,T]} + \|[P, \chi_{<R_0}]u\|_{LE_c^*[0,T]},$$

and so

$$\begin{aligned} \|u\|_{LE^1[0,T]} &\leq \|\chi_{<R_0} u\|_{LE^1[0,T]} + \|\chi_{>R_0} u\|_{LE^1[0,T]} \\ &\lesssim \left\| \langle x \rangle^{-2} u \right\|_{LE[0,T]} + \|Pu\|_{LE_c^*[0,T]} + \|u\|_{LE^1_{R_0 \leq |\cdot| \leq 2R_0}[0,T]}. \end{aligned}$$

The last term is readily estimated via Proposition 2.5, which establishes (3.14).

□

We record the results of our case reductions in the following proposition.

Proposition 3.17. *In order to establish Theorem 1.6, it is sufficient to prove the estimate*

$$\|v\|_{LE^1[0,T]} \lesssim \|v\|_{L_t^2 L_x^2[0,T]} + \|Pv\|_{LE^*[0,T]} \tag{3.15}$$

for v supported in $\{|x| \leq 2R_0\}$ with $v[0] = v[T] = 0$.

Again, this implicit constant is independent of T but will depend on R_0. Using the compact support of v to transition between the weighted and unweighted spaces will inherently generate multiplication by powers of R_0, but this does not matter since the constant in the above may depend on such a parameter.

3.6 Proof of the High Frequency Estimate

Armed with the established case reductions, we will proceed with a proof of Theorem 1.6. Recall that it is equivalent to prove the theorem for the scaled problem.

Proof of Theorem 1.6. In view of Proposition 3.17, it will suffice to prove (3.15) for v supported in $\{|x| < 2R_0\}$ with $v[0] = v[T] = 0$. We can extend v by zero to be defined for $t \in \mathbb{R}$ and vanish for $t \notin (0,T)$. Then,

(3.16)

$$2\mathrm{Im}\left\langle Pv, \left(q^{\mathrm{w}} - \frac{i}{2}m^{\mathrm{w}}\right)v\right\rangle + \frac{i\gamma}{2}\langle [aD_t, m^{\mathrm{w}}]v, v\rangle = \langle i[\Box_g, q^{\mathrm{w}}]v, v\rangle + \gamma\langle (q^{\mathrm{w}}aD_t + aD_t q^{\mathrm{w}})v, v\rangle + \frac{1}{2}\langle (\Box_g m^{\mathrm{w}} + m^{\mathrm{w}}\Box_g)v, v\rangle .$$

The right-hand side of (3.16) can be written as

$$\begin{aligned}\langle i[\Box_g, q^{\mathrm{w}}]v, v\rangle + \gamma\langle (q^{\mathrm{w}}aD_t + aD_t q^{\mathrm{w}})v, v\rangle + \frac{1}{2}\langle (\Box_g m^{\mathrm{w}} + m^{\mathrm{w}}\Box_g)v, v\rangle \\ = \langle (H_p q + 2\gamma\tau a q + mp)^{\mathrm{w}} v, v\rangle + \langle A_0 v, v\rangle ,\end{aligned}$$

where $A_0 \in \Psi^0$. Recall that $2\gamma\tau a = -2is_{skew}$. By choosing $\gamma > 0$ large enough, we can apply Lemma 3.4 to get

$$H_p q - 2is_{skew}q + mp - C\mathbb{1}_{|\xi|\geq\lambda}\langle x\rangle^{-2}(|\xi|^2 + \tau^2) \geq 0, \tag{3.17}$$

where $C > 0$ is the implicit constant in Lemma 3.4. We can readily replace $\mathbb{1}_{|\xi|\geq\lambda}$ with the smooth cutoff $\chi_{|\xi|>\lambda}$. Since the desired estimate is a high frequency estimate, we will first analyze the high frequency components of v. Split $v = v_{>>\lambda} + v_{<<\lambda}$, where

$$\begin{aligned} v_{>>\lambda} &= \chi_{|\xi|+|\tau|>\lambda}(\partial)v, \\ v_{<<\lambda} &= \chi_{|\xi|+|\tau|<\lambda}(\partial)v. \end{aligned}$$

By (3.17), we may apply the sharp Gårding inequality to obtain that

$$\langle (H_p q - 2is_{skew}q + mp)^{\mathrm{w}} v_{>>\lambda}, v_{>>\lambda}\rangle \gtrsim \left\langle \left(\chi_{|\xi|>\lambda}\langle x\rangle^{-2}(|\xi|^2+\tau^2)\right)^{\mathrm{w}} v_{>>\lambda}, v_{>>\lambda}\right\rangle - \|v_{>>\lambda}\|^2_{H^{1/2}_{t,x}} .$$

We remark that the implicit constant may be chosen independently of λ since there is a $\chi_{|\xi|>\lambda}$ cutoff embedded into q and m, and hence differentiation occurring in asymptotic expansion calculations possess coefficients which are either independent of λ or feature inverse powers of λ (one can also entirely ignore the potential λ dependence and argue via Cauchy-Schwarz and Young's inequality for products, although this introduces more parameters to track).

Since $\chi_{|\xi|+|\tau|<\lambda} \in S^{-\infty}$, it follows that

$$\langle (H_p q - 2is_{skew} q + mp)^{\mathrm{w}} v, v\rangle = \langle (H_p q - 2is_{skew} q + mp)^{\mathrm{w}} v_{>>\lambda}, v_{>>\lambda}\rangle + \langle S_0 v, v\rangle ,$$

where $S_0 \in \Psi^{-\infty}$. In particular,

(3.18)

$$\langle (H_p q - 2is_{skew} q + mp)^{\mathrm{w}} v, v\rangle \gtrsim \left\langle \left(\chi_{|\xi|>\lambda} \langle x\rangle^{-2} (|\xi|^2+\tau^2)\right)^{\mathrm{w}} v_{>>\lambda}, v_{>>\lambda}\right\rangle - \|v_{>>\lambda}\|^2_{H^{1/2}_{t,x}} + \langle S_0 v, v\rangle .$$

Using the pseudodifferential composition formula, we compute that

$$\left(\chi_{|\xi|>\lambda}\langle x\rangle^{-2}(|\xi|^2+\tau^2)\right)^{\mathrm{w}} = (\chi_{|\xi|>\lambda}(D_x))^{1/2} D_\alpha \langle x\rangle^{-2} D_\alpha (\chi_{|\xi|>\lambda}(D_x))^{1/2} + A_1,$$

where $A_1 \in \Psi^1$ arises from non-principal terms in the asymptotic expansion of the Moyal product (and the expansion features terms which are either independent of λ or involve inverse powers of λ). Integrating by parts once gives that

$$\begin{aligned} \left\langle \left(\chi_{|\xi|>\lambda}\langle x\rangle^{-2}(|\xi|^2+\tau^2)\right)^{\mathrm{w}} v_{>>\lambda}, v_{>>\lambda}\right\rangle &= \left\| \langle x\rangle^{-1} \partial v_{>\lambda}\right\|^2_{L^2_t L^2_x} + \langle A_1 v_{>>\lambda}, v_{>>\lambda}\rangle \\ &\gtrsim \|\partial v_{>\lambda}\|^2_{LE_{<2R_0}} + \langle A_1 v_{>>\lambda}, v_{>>\lambda}\rangle , \end{aligned} \tag{3.19}$$

where $v_{>\lambda} = \chi_{|\xi|>\lambda}(D_x) v$. One might expect the term $\partial\left((\chi_{|\xi|>\lambda}(D_x))^{1/2}) v_{>>\lambda}\right)$ to appear instead of $\partial v_{>\lambda}$, but it is readily seen that

$$\left(\chi_{|\xi|>\lambda}(|\xi|)\right)^{1/2} \chi_{|\xi|+|\tau|>\lambda}(|(\tau,\xi)|) \approx \chi_{|\xi|>\lambda}(|\xi|)\chi_{|\xi|+|\tau|>\lambda}(|(\tau,\xi)|) = \chi_{|\xi|>\lambda}(|\xi|).$$

In particular, the τ has no effect on the resulting cutoff, and $\chi, \chi^{1/2}$ are both smooth, non-decreasing, and have the same support properties (and only differ on a compact set). For this reason, none of our analysis changes by working with $v_{>\lambda}$, and we will stick with this for notational convenience.

After incorporating (3.19) into (3.18), we have that

$$\langle (H_p q - 2is_{skew}aq + mp)^{\mathrm{w}} v, v\rangle + \langle A_0 v, v\rangle \gtrsim \|\partial v_{>\lambda}\|^2_{LE_{<2R_0}} - \|v_{>>\lambda}\|^2_{H^{1/2}_{t,x}}$$
$$- |\langle A_1 v_{>>\lambda}, v_{>>\lambda}\rangle| - |\langle A_0 v, v\rangle| - |\langle S_0 v, v\rangle| .$$

We will first analyze the term $\langle A_1 v_{>>\lambda}, v_{>>\lambda}\rangle$. Since $A_1 \in \Psi^1$, it is bounded from $H^1_{t,x}$ to $L^2_t L^2_x$ (and the operator norm will yield no positive-power λ contributions due to the previous comment on the asymptotic expansion of the symbol). By using the Schwarz inequality and this mapping property, we have that

$$|\langle A_1 v_{>>\lambda}, v_{>>\lambda}\rangle| \lesssim \|v_{>>\lambda}\|_{H^1_{t,x}} \|v_{>>\lambda}\|_{L^2_t L^2_x} . \tag{3.20}$$

Using Plancherel's theorem in (t, x), the frequency localization, and the compact support of v, we obtain the bounds

$$\|v_{>>\lambda}\|_{H^1_{t,x}} \lesssim \left\|\langle(\tau,\xi)\rangle \chi_{|\xi|+|\tau|>\lambda}\hat{v}\right\|_{L^2_\tau L^2_\xi} \lesssim \|\langle(\tau,\xi)\rangle \hat{v}\|_{L^2_\tau L^2_\xi} = \|v\|_{H^1_{t,x}} \lesssim \|v\|_{LE^1} , \tag{3.21}$$

and

$$\|v_{>>\lambda}\|_{L^2_t L^2_x} \approx \left\|\chi_{|\xi|+|\tau|>\lambda}\hat{v}\right\|_{L^2_\tau L^2_\xi} \lesssim \left\|\frac{|\tau|+|\xi|}{\lambda}\chi_{|\xi|+|\tau|>\lambda}\hat{v}\right\|_{L^2_\tau L^2_\xi} \lesssim \lambda^{-1}\|\partial v\|_{L^2_t L^2_x} \lesssim \lambda^{-1}\|v\|_{LE^1} . \tag{3.22}$$

Applying (3.21) and (3.22) to (3.20) yields that

$$|\langle A_1 v_{>>\lambda}, v_{>>\lambda}\rangle| \lesssim \lambda^{-1}\|v\|^2_{LE^1} .$$

For the term $\|v_{>>\lambda}\|^2_{H^{1/2}_{t,x}}$, note that

$$\|v_{>>\lambda}\|^2_{H^{1/2}_{t,x}} \lesssim \left\|\langle(\tau,\xi)\rangle^{1/2} \chi_{|\xi|+|\tau|>\lambda}\hat{v}\right\|^2_{L^2_\tau L^2_\xi} = \left\|\langle(\tau,\xi)\rangle^{-1/2}\langle(\tau,\xi)\rangle \chi_{|\xi|+|\tau|>\lambda}\hat{v}\right\|^2_{L^2_\tau L^2_\xi}$$
$$\lesssim \lambda^{-1}\left\|\langle(\tau,\xi)\rangle \chi_{|\xi|+|\tau|>\lambda}\hat{v}\right\|^2_{L^2_\tau L^2_\xi} \lesssim \lambda^{-1}\|v\|^2_{LE^1} .$$

For the $\langle A_0 v, v\rangle$ term, we can use L^2-boundedness and the compact support of v to get

$$|\langle A_0 v, v\rangle| \leq \|A_0 v\|_{L_t^2 L_x^2} \|v\|_{L_t^2 L_x^2} \lesssim C(\lambda) \|v\|_{L_t^2 L_x^2}^2 . \tag{3.23}$$

While this bound is λ-dependent, such terms appear on the *upper bound side* of the desired inequality, and hence can depend on λ in an arbitrary manner (as opposed to the LE^1 terms which need an inverse power of λ for bootstrapping). The meaning of $C(\lambda)$ will change fluidly, just as one continuously re-notates a potentially-changing constant by C when calculating successive inequalities.

The smoothing term $\langle S_0 v, v\rangle$ can be bounded in the same way as $\langle A_0 v, v\rangle$ (in particular, $S_0 \in \Psi^0$). Thus, we have the lower bound

$$\langle (H_p q - 2is_{skew} q + mp)^{\mathrm{w}} v, v\rangle + \langle A_0 v, v\rangle \gtrsim \|\partial v_{>\lambda}\|_{LE_{<2R_0}}^2 - C(\lambda) \|v\|_{L_t^2 L_x^2}^2 - \lambda^{-1} \|v\|_{LE^1}^2 . \tag{3.24}$$

Next, we look at the left-hand side of (3.16). Since $[aD_t, m^{\mathrm{w}}] \in \Psi^0$, performing the same work as in (3.23) provides that

$$\left| \frac{i\gamma}{2} \langle [aD_t, m^{\mathrm{w}}]v, v\rangle \right| \lesssim C(\lambda) \|v\|_{L_t^2 L_x^2}^2 . \tag{3.25}$$

For remaining term on the left-hand side of (3.16), we split v into high and low frequency components once again to get that

$$2\mathrm{Im} \left\langle Pv, \left(q^{\mathrm{w}} - \frac{i}{2} m^{\mathrm{w}}\right) v \right\rangle = 2\mathrm{Im} \left\langle Pv, \left(q^{\mathrm{w}} - \frac{i}{2} m^{\mathrm{w}}\right) v_{>>\lambda} \right\rangle + \langle S_1 v, v\rangle ,$$

where $S_1 \in \Psi^{-\infty}$. We have already demonstrated how to bound smoothing operator terms. For the other (primary) piece, we apply the Schwarz inequality, use the ΨDO mapping properties of $q^{\mathrm{w}} \in \Psi^1$ and $m^{\mathrm{w}} \in \Psi^0$, and leverage the compact support of v (just as performed previously) to get

$$\left| 2\mathrm{Im} \left\langle Pv, \left(q^{\mathrm{w}} - \frac{i}{2} m^{\mathrm{w}}\right) v_{>>\lambda} \right\rangle \right| \lesssim C(\lambda) \|Pv\|_{L_t^2 L_x^2} \|v\|_{LE^1} \lesssim C(\lambda) \|Pv\|_{LE_c^*} \|v\|_{LE^1} . \tag{3.26}$$

Putting (3.16), (3.24), (3.25), and (3.26) together, we obtain that

$$\|\partial v_{>\lambda}\|_{LE_{<2R_0}} \lesssim C(\lambda)\left(\|Pv\|_{LE^*}^{1/2}\|v\|_{LE^1}^{1/2} + \|v\|_{L_t^2L_x^2}\right) + \lambda^{-1/2}\|v\|_{LE^1}.$$

Completing the $LE^1_{<2R_0}$ norm on the left-hand side of the above,

$$\|v_{>\lambda}\|_{LE^1_{<2R_0}} \lesssim C(\lambda)\left(\|Pv\|_{LE^*}^{1/2}\|v\|_{LE^1}^{1/2} + \|v\|_{L_t^2L_x^2}\right) + \lambda^{-1/2}\|v\|_{LE^1} + \left\|\langle x\rangle^{-1} v_{>\lambda}\right\|_{LE}.$$

We note that

$$\left\|\langle x\rangle^{-1} v_{>\lambda}\right\|_{LE} \lesssim \|v\|_{L_t^2L_x^2},$$

once again using Plancherel's theorem. Thus,

$$\|v_{>\lambda}\|_{LE^1_{<2R_0}} \lesssim C(\lambda)\left(\|Pv\|_{LE^*}^{1/2}\|v\|_{LE^1}^{1/2} + \|v\|_{L_t^2L_x^2}\right) + \lambda^{-1/2}\|v\|_{LE^1}. \tag{3.27}$$

This establishes an estimate on the high frequencies. We must add in the lower frequencies to the left-hand side. That is, we must add $\|v_{<\lambda}\|_{LE^1_{<2R_0}}$ to both sides. First, we get the bound

$$\left\|\langle x\rangle^{-1} v_{<\lambda}\right\|_{LE} \lesssim \|v\|_{L_t^2L_x^2}$$

via Plancherel's theorem. For the term $\|\partial v_{<\lambda}\|_{LE}$, we write

$$v_{<\lambda} = v_{<>\sigma\lambda} + v_{<<\sigma\lambda},$$

where

$$v_{<>\sigma\lambda} = \chi_{|\xi|<\lambda}(D_x)\chi_{|\tau|>\sigma\lambda}(D_t)v,$$
$$v_{<<\sigma\lambda} = \chi_{|\xi|<\lambda}(D_x)\chi_{|\tau|<\sigma\lambda}(D_t)v,$$

and $\sigma \gg 1$ will be chosen later (and does not denote the same σ as used in the construction of the escape function). Applying Plancherel's theorem, frequency localization, and the compact support of v once again yields

$$\|\partial v_{<<\sigma\lambda}\|_{LE} \lesssim \left\|(|\tau|+|\xi|)\chi_{|\xi|<\lambda}\chi_{|\tau|<\sigma\lambda}\hat{v}\right\|_{L^2_\tau L^2_\xi} \lesssim \sigma\lambda \|v\|_{L^2_t L^2_x}.$$

For $v_{<>\sigma\lambda}$, we compute that

$$\|\partial v_{<>\sigma\lambda}\|_{LE} \lesssim \left\|(|\tau|+|\xi|)\chi_{|\xi|<\lambda}\chi_{|\tau|>\sigma\lambda}\hat{v}\right\|_{L^2_\tau L^2_\xi} \lesssim \lambda \|v\|_{L^2_t L^2_x} + (\sigma\lambda)^{-1}\left\|(\partial_t^2 v)_{<>\sigma\lambda}\right\|_{L^2_t L^2_x}. \tag{3.28}$$

For the last term on the right, we utilize the expression for Pv to write

$$\begin{aligned}\left\|(\partial_t^2 v)_{<>\sigma\lambda}\right\|_{L^2_t L^2_x} \lesssim \|(Pv)_{<>\sigma\lambda}\|_{L^2_t L^2_x} &+ \left\|\left((g^{0j}D_j + D_j g^{0j})D_t v\right)_{<>\sigma\lambda}\right\|_{L^2_t L^2_x} \\ &+ \left\|(D_i g^{ij} D_j v)_{<>\sigma\lambda}\right\|_{L^2_t L^2_x} + \|(aD_t v)_{<>\sigma\lambda}\|_{L^2_t L^2_x}.\end{aligned} \tag{3.29}$$

One can readily check that

$$\|(Pv)_{<>\sigma\lambda}\|_{L^2_t L^2_x} \lesssim \|Pv\|_{LE^*}, \tag{3.30}$$

and

$$\|(aD_t v)_{<>\sigma\lambda}\|_{L^2_t L^2_x} \lesssim \|\partial v\|_{LE}. \tag{3.31}$$

For the other terms, we note that as *functions*, one has that $g^{\alpha j}, D_j g^{\alpha j} \in S^0$ for all $\alpha \in \{0,1,2,3\}$ and $j \in \{1,2,3\}$, and so

$$[\chi_{|\xi|<\lambda}(D_x)\chi_{|\tau|>\sigma\lambda}(D_t), g^{\alpha j}] \in \Psi^{-1},$$
$$[\chi_{|\xi|<\lambda}(D_x)\chi_{|\tau|>\sigma\lambda}(D_t), D_j g^{\alpha j}] \in \Psi^{-1}.$$

In particular, the above two operators are bounded on $L^2_t L^2_x$. Pairing this with the fact that Fourier multipliers commute, we have that

$$(3.32)\quad \left\|\left((g^{0j}D_j + D_jg^{0j})D_tv\right)_{<>\sigma\lambda}\right\|_{L^2_tL^2_x} \lesssim \left\|(D_jg^{0j})\,(D_tv)_{<>\sigma\lambda}\right\|_{L^2_tL^2_x} + \left\|g^{0j}\,(D_jD_tv)_{<>\sigma\lambda}\right\|_{L^2_tL^2_x}$$
$$+ \left\|[\chi_{|\xi|<\lambda}(D_x)\chi_{|\tau|>\sigma\lambda}(D_t), (D_jg^{0j})]D_tv\right\|_{L^2_tL^2_x}$$
$$+ \left\|[\chi_{|\xi|<\lambda}(D_x)\chi_{|\tau|>\sigma\lambda}(D_t), g^{0j}]D_jD_tv\right\|_{L^2_tL^2_x}$$
$$\lesssim \lambda\,\|\partial v\|_{LE} + C(\lambda)\,\|v\|_{L^2_tL^2_x}\,,$$

and

$$(3.33)\quad \left\|(D_ig^{ij}D_jv)_{<>\sigma\lambda}\right\|_{L^2_tL^2_x} \lesssim \left\|(D_ig^{ij})(D_jv)_{<>\sigma\lambda}\right\|_{L^2_tL^2_x} + \left\|g^{ij}(D_iD_jv)_{<>\sigma\lambda}\right\|_{L^2_tL^2_x}$$
$$+ \left\|([\chi_{|\xi|<\lambda}(D_x)\chi_{|\tau|>\sigma\lambda}(D_t), (D_ig^{ij})](D_jv)_{<>\sigma\lambda}\right\|_{L^2_tL^2_x}$$
$$+ \left\|([\chi_{|\xi|<\lambda}(D_x)\beta_{|\tau|\geq\sigma\lambda}(D_t), g^{ij}](D_iD_jv)_{<>\sigma\lambda}\right\|_{L^2_tL^2_x}$$
$$\lesssim C(\lambda)\,\|v\|_{L^2_tL^2_x}\,.$$

Applying (3.30)-(3.33) to (3.29) gives that

$$\left\|(\partial_t^2 v)_{<>\sigma\lambda}\right\|_{L^2_tL^2_x} \lesssim C(\lambda)\,\|v\|_{L^2_tL^2_x} + \lambda\,\|\partial v\|_{LE} + \|Pv\|_{LE^*}\,.$$

Plugging the resulting estimate into (3.28) implies that

$$\|\partial v_{<>\sigma\lambda}\|_{LE} \lesssim C(\lambda)\,\|v\|_{L^2_tL^2_x} + (\sigma\lambda)^{-1}\,\|Pv\|_{LE^*} + \sigma^{-1}\,\|\partial v\|_{LE}\,.$$

Thus, the full low frequency contribution yields

$$(3.34)\quad \|\partial v_{<\lambda}\|_{LE} \lesssim \max\{C(\lambda), \sigma\lambda\}\,\|v\|_{L^2_tL^2_x} + (\sigma\lambda)^{-1}\,\|Pv\|_{LE^*} + \sigma^{-1}\,\|\partial v\|_{LE}$$
$$\lesssim \max\{C(\lambda), \sigma\lambda\}\,\|v\|_{L^2_tL^2_x} + (\sigma\lambda)^{-1}\,\|Pv\|_{LE^*} + \sigma^{-1}\,\|v\|_{LE^1}\,.$$

Now, we can combine the high frequency work (3.27) with the low frequency work (3.34) and apply Young's inequality for products with parameter $\delta > 0$ to obtain that

$$\begin{aligned}
&\|v\|_{LE^1_{<2R_0}} \\
&\lesssim C(\lambda)\,\|Pv\|_{LE^*}^{1/2}\,\|v\|_{LE^1}^{1/2} + \max\{C(\lambda),\sigma\lambda\}\,\|v\|_{L^2_tL^2_x} + (\sigma\lambda)^{-1}\,\|Pv\|_{LE^*} + \left(\sigma^{-1}+\lambda^{-1/2}\right)\|v\|_{LE^1} \\
&\lesssim \max\{C(\lambda),\sigma\lambda\}\,\|v\|_{L^2_tL^2_x} + \left([C(\lambda)]^2\delta^{-1} + (\sigma\lambda)^{-1}\right)\|Pv\|_{LE^*} + \left(\delta+\sigma^{-1}+\lambda^{-1/2}\right)\|v\|_{LE^1}
\end{aligned}$$

Due to the support of v in x, we know that $\|v\|_{LE^1_{<2R_0}} = \|v\|_{LE^1}$. Picking δ sufficiently small and λ, σ sufficiently large (all of which will depend on R_0) allows us to absorb the $\|v\|_{LE^1}$ term on the right-hand side into the left-hand side, providing (3.15) and completing the proof. □

3.7 An Application to High Energy Resolvent Estimates

Here, we establish a direct link between our high frequency estimate (1.6) and a high energy resolvent bound. Analogous to the local energy spaces, we define the spaces $\mathcal{LE}, \mathcal{LE}^1, \mathcal{LE}^*$ when the time variable is fixed (and there is no time derivative involved in the norms, either). We will also require spaces which allow us to track dependence on the spectral parameter ω, namely

$$\begin{aligned}
\mathcal{LE}^1_\omega &= \mathcal{LE}^1 \cap |\omega|^{-1}\mathcal{LE}, \\
\dot{H}^1_\omega &= \dot{H}^1 \cap |\omega|^{-1}L^2.
\end{aligned}$$

These spaces are equipped with norms

$$\begin{aligned}
\|u\|_{\mathcal{LE}^1_\omega} &= \|u\|_{\mathcal{LE}^1} + |\omega|\,\|u\|_{\mathcal{LE}}\,, \\
\|u\|_{\dot{H}^1_\omega} &= \|u\|_{\dot{H}^1} + |\omega|\,\|u\|_{L^2}\,,
\end{aligned}$$

respectively. Now, we will define the resolvent. Consider $Pu = 0$, where (as usual)

$$P = D_\alpha g^{\alpha\beta} D_\beta + iaD_t.$$

One arrives at the *stationary problem* by looking for solutions of the form $u(t,x) = e^{i\omega t}u_\omega(x)$ (equivalently, one replaces D_t by ω). This generates the equation

$$P_\omega u_\omega = 0,$$

where

$$P_\omega = g^{00}\omega^2 + (g^{0j}D_j + D_j g^{0j} + ia)\omega + D_i g^{ij} D_j.$$

The *resolvent* R_ω is defined as the inverse of P_ω when such an inverse exists. More explicitly, if we consider the homogeneous Cauchy problem

$$Pu = 0, \qquad u(0) = 0, \qquad -g^{00}\partial_t u(0) = f,$$

then we can formally define R_ω via the Fourier-Laplace transform of u:

$$R_\omega f = \int_0^\infty e^{-i\omega t} u(t)\, dt, \qquad\qquad \omega \in \mathcal{H} := \{z : \mathrm{Im}\ z < 0\}.$$

One can check via integration by parts that both definitions of R_ω are consistent. In the subsequent work, we will take f to be in either L^2 or $\mathcal{LE}^*$, and it will be clear from context which is the case.

An immediate consequence of Proposition 2.1 is the estimate

$$\|\partial u(t)\|_{L^2} \lesssim \|f\|_{L^2}\,. \tag{3.35}$$

Using (3.35) and the Minkowski integral inequality, we obtain that

$$\|R_\omega f\|_{\dot H^1} \le \int_0^\infty e^{\mathrm{Im}\,\omega t} \|\nabla u(t,\cdot)\|_{L^2}\, dt \le \int_0^\infty e^{\mathrm{Im}\,\omega t} \|f\|_{L^2}\, dt \lesssim \frac{1}{|\,\mathrm{Im}\,\omega|} \|f\|_{L^2}\,, \qquad \omega \in \mathcal{H}. \tag{3.36}$$

Meanwhile, integrating by parts once provides that

$$\omega R_\omega f = -i \int_0^\infty e^{-i\omega t}\partial_t u(t)\, dt, \qquad \mathrm{Im}\,\omega < 0.$$

Taking the L^2 norm and performing the same work as in (3.36) yields an identical upper bound. Combining these estimates together gives the inequality

$$\|R_\omega f\|_{\dot{H}^1_\omega} \lesssim \frac{1}{|\operatorname{Im}\omega|} \|f\|_{L^2}, \qquad \omega \in \mathcal{H}.$$

This estimate shows that P_ω has no eigenfunctions with corresponding eigenvalues in the lower half-plane. We record this work in the following proposition.

Proposition 3.18 (Uniform Energy Resolvent Bound). *For $\omega \in \mathcal{H}$, the resolvent is an analytic family of bounded operators $R_\omega : L^2 \to \dot{H}^1_\omega$ satisfying*

$$\|R_\omega\|_{L^2 \to \dot{H}^1_\omega} \lesssim |\text{Im } \omega|^{-1}, \qquad \omega \in \mathcal{H}.$$

In [27], it is proven that local energy decay for stationary, asymptotically flat wave operators is equivalent to the *local energy resolvent bound*

$$\|R_\omega\|_{\mathcal{LE}^* \to \mathcal{LE}^1_\omega} \lesssim 1, \qquad \omega \in \mathcal{H}. \tag{3.37}$$

One can readily use Proposition 3.18 to obtain this estimate for $\operatorname{Im}\omega \lesssim -1$.

Proposition 3.19. *If P is a stationary, asymptotically flat damped wave operator, then the resolvent R_ω satisfies the bound*

$$\|R_\omega\|_{\mathcal{LE}^* \to \mathcal{LE}^1_\omega} \lesssim 1 \qquad \operatorname{Im}\omega \lesssim -1.$$

Proof. For $-\operatorname{Im}\omega \gtrsim 1$, we can obtain the desired bound straightforwardly by applying Proposition 3.18:

$$\begin{aligned}
\|R_\omega f\|_{\mathcal{LE}^1_\omega} &= \|\nabla R_\omega f\|_{\mathcal{LE}} + \left\|\langle x\rangle^{-1} R_\omega f\right\|_{\mathcal{LE}} + |\omega| \, \|R_\omega f\|_{\mathcal{LE}} \\
&\lesssim \|R_\omega f\|_{\dot{H}^1} + |\omega| \, \|R_\omega f\|_{L^2_x} \\
&= \|R_\omega f\|_{\dot{H}^1_\omega} \\
&\lesssim \frac{1}{|\operatorname{Im}\omega|} \|f\|_{L^2} \\
&\lesssim \|f\|_{\mathcal{LE}^*}.
\end{aligned}$$

□

We can use Theorem 1.6 to obtain (3.37) for large frequencies close to the real axis.

Proposition 3.20. *Let P be a stationary, asymptotically flat damped wave operator satisfying the geometric control condition, and suppose that ∂_t is uniformly time-like. Then, the resolvent R_ω satisfies the bound*

$$\|R_\omega\|_{\mathcal{LE}^* \to \mathcal{LE}^1_\omega} \lesssim 1 \qquad |\omega| \gg 1,\ -1 \lesssim \operatorname{Im}\omega < 0.$$

We proceed as in [27].

Proof. Let u solve $P_\omega u = f$, and call $v = e^{i\omega t}u$. Then, v solves $Pv = g$, where $g = e^{i\omega t}f$. We will apply the high frequency estimate to the interval $[-T, 0]$. More precisely, if we call $\tilde{v}(t,x) = v(t-T, x)$, then this solves $P\tilde{v} = \tilde{g}$, where $\tilde{g} = e^{i\omega(t-T)}f$. Applying Theorem 1.6 to $\tilde{v}$ provides

$$\|\tilde{v}\|_{LE^1[0,T]} + \|\partial\tilde{v}\|_{L^\infty_t L^2_x[0,T]} \lesssim \|\partial\tilde{v}(0)\|_{L^2} + \left\|\langle x\rangle^{-2}\tilde{v}\right\|_{LE[0,T]} + \|\tilde{g}\|_{LE^* + L^1_t L^2_x[0,T]}.$$

We immediately calculate that

$$\begin{aligned}
\|\tilde{v}\|_{LE^1[0,T]} &= \left(\frac{e^{2T\operatorname{Im}\omega} - 1}{2\operatorname{Im}\omega}\right)^{1/2} \|u\|_{\mathcal{LE}^1_\omega} \\
\|\partial\tilde{v}\|_{L^\infty_t L^2_x[0,T]} &\approx \|u\|_{\dot{H}^1_\omega} \\
\|\partial\tilde{v}(0)\|_{L^2} &\approx e^{T\operatorname{Im}\omega}\|u\|_{\dot{H}^1_\omega} \\
\left\|\langle x\rangle^{-2}\tilde{v}\right\|_{LE[0,T]} &= \left(\frac{e^{2T\operatorname{Im}\omega} - 1}{2\operatorname{Im}\omega}\right)^{1/2} \left\|\langle x\rangle^{-2}u\right\|_{\mathcal{LE}} \\
\|\tilde{g}\|_{LE^* + L^1_t L^2_x[0,T]} &= \|f\|_{\left(\frac{\exp(2T\operatorname{Im}\omega)-1}{2\operatorname{Im}\omega}\right)^{-1/2}\mathcal{LE}^* + \left(\frac{\exp(T\operatorname{Im}\omega)-1}{\operatorname{Im}\omega}\right)^{-1}L^2}.
\end{aligned}$$

Plugging these calculations into the high frequency bound and taking the limit as $T \to \infty$ yields

$$\|u\|_{\mathcal{LE}^1_\omega} + |\operatorname{Im}\omega|^{1/2}\|u\|_{\dot{H}^1_\omega} \lesssim \left\|\langle x\rangle^{-2}u\right\|_{\mathcal{LE}} + \|f\|_{\mathcal{LE}^* + |\operatorname{Im}\omega|^{1/2}L^2}.$$

For sufficiently large ω, the first term on the right absorbs into the first term on the left, which implies the desired bound. □

Combining Proposition 3.19 with Proposition 3.20 immediately establishes (3.37) for all ω in the lower half-plane outside of a relatively compact set near the real line, which is the content of the following theorem.

Theorem 3.21. *Let P be a stationary, asymptotically flat damped wave operator satisfying the geometric control condition, and suppose that ∂_t is uniformly time-like. Then, the resolvent R_ω satisfies the bound*

$$\|R_\omega\|_{\mathcal{LE}^* \to \mathcal{LE}^1_\omega} \lesssim 1, \qquad \omega \in \mathcal{H} \setminus \{\zeta \in \mathcal{H} : |\zeta| \lesssim 1\}.$$

In [27], the estimate is proven in the rest of the lower half-plane using a low frequency estimate for ω close to zero and a limiting absorption argument for the remaining frequencies. Although this work likely holds in our context due to possessing the same frequency estimates, we will omit it here (see [27] for this work); we only included the high frequency resolvent estimate here since our high frequency estimate was the primary focus of this work.

CHAPTER 4

Medium Frequency Analysis

4.1 Introduction

Here, we establish weighted estimates which imply local energy decay for solutions supported at any range of time frequencies bounded away from both zero and infinity. These will be rooted in *Carleman estimates*, which are weighted $L^2_t L^2_x$ estimates where the weight is (in principle) convex. Such estimates were originally studied to establish unique continuation results, which can be used to prove e.g. the absence of embedded eigenvalues for certain classes of problems (see [20] and the references therein). The Carleman estimates that we desire take the general form

$$\|\rho_0 e^{\varphi} u\|_{L^2_t L^2_x} + \|\rho_1 e^{\varphi} \partial u\|_{L^2_t L^2_x} \lesssim \|e^{\varphi} P u\|_{L^2_t L^2_x},$$

for appropriate integration weights ρ_0, ρ_1 and Carleman weight φ. The constants in our inequalities will depend on the parameter $\mathbf{c}$ introduced in Section 1.2, but they will (and must) be independent of the parameters in φ (our weights will be radial). The Carleman weights which we will use are constructed in e.g. [5], [20], [38]. Our approach closely follows that of [27] and [5], and we will prove the same general results. As opposed to working on the symbol side (as done in [27]), we will work directly on the differential operator side (as done in [5]). This has the benefit of illuminating explicit error terms, which one would pick up from pseudodifferential calculus, at the expense of more tedious calculations and terms to track. The work [5] studied small time-dependent perturbations of asymptotically Euclidean metrics, hence our work will feature some deviation. We will not assume that our metric is stationary for this chapter.

4.2 Exterior Carleman Estimates

Our first class of Carleman estimates apply in the exterior region $\{|x| > R_0\}$. Since $a \equiv 0$ when $|x| > R_0$, our operator reduces to $P = D_\alpha g^{\alpha\beta} D_\beta$ here. While the results in [27] (Propositions 5.1 and 5.2) apply directly in this setting, we will re-establish them here. This is also similar to the work on absence of embedded eigenvalues present in [20]. For the remainder of this section, denote $s = \ln r$.

Proposition 4.1. *Let P be an asymptotically flat damped wave operator and φ be a convex function satisfying*

$$\lambda \lesssim \varphi'(s), \qquad \lambda \lesssim \varphi''(s) \leq \frac{1}{2}\varphi'(s) \lesssim \varphi''(s), \qquad |\varphi'''(s)| \ll \varphi'(s),$$

where $\lambda \gg 1$. Then, for all $u \in \mathcal{S}(\mathbb{R}^4)$ with $\operatorname{supp} u \subset \{r > R_0\}$*, we have the estimate*

$$\left\| r^{-1}(1+\varphi'')^{1/2} e^{\varphi} \left(r^{-1}(1+\varphi')u, \nabla u\right) \right\|_{L_t^2 L_x^2} + \left\| r^{-1}(1+\varphi')^{1/2} e^{\varphi} \partial_t u \right\|_{L_t^2 L_x^2} \lesssim \|e^{\varphi} P u\|_{L_t^2 L_x^2}. \tag{4.1}$$

The construction of the weight can be found in [20]. We remark that, in the exterior regime, dominant terms involve larger powers of r and derivatives of the weight φ.

Proof. First, we conjugate P by e^{φ} to form $P_\varphi = e^{\varphi} P e^{-\varphi}$. If we call $v = e^{\varphi} u$, then (4.1) becomes

$$\left\| r^{-1}(1+\varphi'')^{1/2} \left(r^{-1}(1+\varphi')v, \nabla v - \frac{x}{r^2}\varphi' v\right) \right\|_{L_t^2 L_x^2} + \left\| r^{-1}(1+\varphi')^{1/2} \partial_t v \right\|_{L_t^2 L_x^2} \lesssim \|P_\varphi v\|_{L_t^2 L_x^2}.$$

Hence, it suffices to prove the estimate

$$\left\| r^{-1}(1+\varphi'')^{1/2} \left(r^{-1}(1+\varphi')v, \nabla v\right) \right\|_{L_t^2 L_x^2} + \left\| r^{-1}(1+\varphi')^{1/2} \partial_t v \right\|_{L_t^2 L_x^2} \lesssim \|P_\varphi v\|_{L_t^2 L_x^2}. \tag{4.2}$$

Next, we decompose P_φ into the sum of its self-adjoint and skew-adjoint parts

$$P_\varphi = P_\varphi^r + P_\varphi^i,$$

respectively. One can compute explicitly that

$$P_\varphi^r = D_\alpha g^{\alpha\beta} D_\beta - \varphi_i g^{ij} \varphi_j,$$
$$P_\varphi^i = i D_\alpha g^{\alpha j} \varphi_j + i \varphi_j g^{j\alpha} D_\alpha,$$

where we are using the notation $\varphi_j := \partial_j \varphi$. Since these operators provide a decomposition of P_φ

into its self-adjoint and skew-adjoint parts, it follows that

$$\|P_\varphi v\|_{L_t^2 L_x^2} = \left\|P_\varphi^r v\right\|_{L_t^2 L_x^2} + \left\|P_\varphi^i v\right\|_{L_t^2 L_x^2} + \left\langle [P_\varphi^r, P_\varphi^i] v, v\right\rangle.$$

We see from here that it will be sufficient to perform a positive commutator argument.

We will prove that

$$\left\|r^{-1}(1+\varphi'')^{1/2}\left(r^{-1}(1+\varphi')v, \nabla v\right)\right\|_{L_t^2 L_x^2}^2 + \left\|r^{-1}(1+\varphi')^{1/2}\partial_t v\right\|_{L_t^2 L_x^2}^2 \lesssim \langle [P_\varphi^r, P_\varphi^i]v, v\rangle$$
$$+ 2\left\|(\varphi')^{-1/2} P_\varphi^i v\right\|_{L_t^2 L_x^2}^2 + \left\langle \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) v, P_\varphi^r v\right\rangle. \tag{4.3}$$

Since $\varphi' \gg 1$ and

$$\left|\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right|^2 = \frac{1}{4} r^{-4}(\varphi'')^2 - 4r^{-4}\varphi''\varphi' + 16 r^{-4}(\varphi')^2 \ll r^{-4}(1+\varphi'')(1+\varphi')^2,$$

applying the Schwarz inequality and Young's inequality for products to (4.3) implies (4.2).

It remains to prove (4.3). We will compute each term on the right-hand side separately, going from right to left. It will be beneficial to record that

$$\begin{aligned}
\varphi_j &= \frac{x_j}{r^2}\varphi', \\
\varphi_{ij} &= \frac{r^2\delta_{ij}\varphi' + x_i x_j\left(\varphi'' - 2\varphi'\right)}{r^4} \\
\varphi_{ijk} &= \frac{(2x_k\delta_{ij}\varphi' + r^2\delta_{ij}x_k r^{-2}\varphi'')r^4 - 4r^2 x_k r^2 \delta_{ij}\varphi'}{r^8} \\
&\quad + \frac{\left((\delta_{ik}x_j + \delta_{jk}x_i)(\varphi'' - 2\varphi') + r^2 x_i x_j x_k(\varphi''' - 2\varphi'')\right) - 4r^2 x_i x_j x_k(\varphi'' - 2\varphi')}{r^8} \\
&= \mathcal{O}(r^{-3}\left(\varphi' + \varphi'' + \varphi'''\right)).
\end{aligned}$$

In particular,

$$\sum_j \varphi_{jj} = \frac{\varphi' + \varphi''}{r^2}.$$

First, we integrate by parts to obtain that

$$
\begin{aligned}
&\left\langle \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) v, P_\varphi^r v \right\rangle \\
&= \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) g^{\alpha\beta} D_\alpha v \overline{D_\beta v}\, dxdt \\
&\quad + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} g^{\beta j} D_j \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) v \overline{D_\beta v}\, dxdt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) \varphi_i g^{ij} \varphi_j |v|^2\, dxdt \\
&= \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) g^{\alpha\beta} \partial_\alpha v \overline{\partial_\beta v}\, dxdt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} g^{\beta j} \partial_j \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) v \overline{\partial_\beta v}\, dxdt \\
&\quad + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) \varphi_i g^{ij} \varphi_j |v|^2\, dxdt.
\end{aligned}
$$

Using Young's inequality for products on the terms involving only one derivative of v gives

$$
\begin{aligned}
&\left\langle \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) v, P_\varphi^r v \right\rangle \\
&\gtrsim \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) g^{\alpha\beta} \partial_\alpha v \overline{\partial_\beta v}\, dxdt - \sum_{\beta=0}^{3} \frac{1}{2} \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left(g^{\beta j} r \partial_j \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)\right)^2 |v|^2\, dxdt \\
&\qquad - \frac{1}{2} \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{|\partial v|^2}{r^2}\, dxdt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) \varphi_i g^{ij} \varphi_j |v|^2\, dxdt.
\end{aligned}
$$

Since $\operatorname{supp} v \subset \{|x| > R_0\}$, we are integrating over the spatial region where g is a small AF perturbation of m, i.e. $\|g - m\|_{AF_{>R_0}} \ll 1$. Writing $g^{\alpha\beta} = (g^{\alpha\beta} - m^{\alpha\beta}) + m^{\alpha\beta}$ and using asymptotic flatness, we get the lower bound

$$
\begin{aligned}
&\left\langle \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) v, P^r_\varphi v \right\rangle \\
&\gtrsim \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)(g^{\alpha\beta} - m^{\alpha\beta})\partial_\alpha v \overline{\partial_\beta v}\, dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{4\varphi'}{r^2} - \frac{\varphi''}{2r^2}\right)|\partial_t v|^2\, dxdt \\
&\quad + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)|\nabla_x v|^2\, dxdt - \frac{1}{2}\sum_{\beta=0}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left((g^{\beta j} - m^{\beta j}) r\partial_j \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)\right)^2 |v|^2\, dxdt \\
&\quad - \sum_{\beta=0}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} (g^{\beta j} - m^{\beta j}) m^{\beta j} \left(r\partial_j \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)\right)^2 |v|^2\, dxdt \\
&\quad - \frac{1}{2}\sum_{j=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(r\partial_j \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)\right)^2 |v|^2\, dxdt \\
&\quad - \frac{1}{2}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{|\partial v|^2}{r^2}\, dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)\varphi_i (g^{ij} - m^{ij})\varphi_j |v|^2\, dxdt \\
&\quad + \sum_{j=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{4\varphi'}{r^2} - \frac{\varphi''}{2r^2}\right)\varphi_j^2 |v|^2\, dxdt \\
&\gtrsim \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{4\varphi'}{r^2} - \frac{\varphi''}{2r^2}\right)|\partial_t v|^2\, dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)|\nabla_x v|^2\, dxdt \\
&\quad - \frac{1}{2}\sum_{j=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(r\partial_j \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)\right)^2 |v|^2\, dxdt \\
&\quad - \frac{1}{2}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{|\partial v|^2}{r^2}\, dxdt + \sum_{j=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{4\varphi'}{r^2} - \frac{\varphi''}{2r^2}\right)\varphi_j^2 |v|^2\, dxdt \\
&\quad - \|g - m\|_{AF_{>R_0}} \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{\varphi'}{r^2}|\partial v|^2\, dxdt - \|g - m\|_{AF_{>R_0}} \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{(\varphi')^3}{r^4}|v|^2\, dxdt.
\end{aligned}
$$

Notice that

$$
\sum_{j=1}^{3}\left(\frac{4\varphi'}{r^2} - \frac{\varphi''}{2r^2}\right)\varphi_j^2 = \sum_{j=1}^{3}\left(\frac{4\varphi'}{r^2} - \frac{\varphi''}{2r^2}\right)\frac{x_j^2}{r^2}\frac{(\varphi')^2}{r^2} = \frac{4(\varphi')^3}{r^4} - \frac{\varphi''(\varphi')^2}{2r^4},
$$

and

$$
\sum_{j=1}^{3}\left(r\partial_j \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)\right)^2 = \sum_{j=1}^{3} r^2 \left(-5\frac{x_j\varphi''}{r^4} + \frac{1}{2}\frac{x_j\varphi'''}{r^4} + 8\frac{x_j\varphi'}{r^4}\right)^2.
$$

In view of the conditions on φ (in particular, the largeness of φ'), the latter term is negligible in comparison to the former for large enough λ. We will hold on to the lower-order term for now.

To summarize our analysis thus far, we have shown that

$$
\begin{aligned}
(4.4)\quad &\left\langle \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right) v, P_\varphi^r v \right\rangle \\
&\gtrsim \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{4\varphi'}{r^2} - \frac{\varphi''}{2r^2} - \frac{1}{2r^2}\right) |\partial_t v|^2 \, dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2} - \frac{1}{2r^2}\right) |\nabla_x v|^2 \, dxdt \\
&\quad + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{4(\varphi')^3}{r^4} - \frac{\varphi''(\varphi')^2}{2r^4}\right) |v|^2 \, dxdt - \frac{1}{2}\sum_{j=1}^{3} \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(r\partial_j \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)\right)^2 |v|^2 \, dxdt \\
&\quad - \|g - m\|_{AF_{>R_0}} \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{\varphi'}{r^2} |\partial v|^2 \, dxdt - \|g - m\|_{AF_{>R_0}} \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{(\varphi')^3}{r^4} |v|^2 \, dxdt,
\end{aligned}
$$

where

$$
\frac{1}{2}\sum_{j=1}^{3} \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(r\partial_j \left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)\right)^2 |v|^2 \, dxdt
$$

is lower-order than

$$
\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{4(\varphi')^3}{r^4} - \frac{\varphi''(\varphi')^2}{2r^4}\right) |v|^2 \, dxdt
$$

in λ due to conditions on the derivatives of φ.

Continuing our analysis of the terms on the right-hand side of (4.3), we calculate that

$$
\begin{aligned}
2\left\|(\varphi')^{-1/2} P_\varphi^i v\right\|_{L_t^2 L_x^2}^2 &= 2 \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} (\varphi')^{-1} \left|(2i\varphi_j g^{j\alpha} D_\alpha + iD_\alpha(g^{\alpha j}\varphi_j))v\right|^2 \, dxdt \\
&= 8 \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} (\varphi')^{-1} |\varphi_j g^{j\alpha}\partial_\alpha v|^2 \, dxdt + 2 \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} (\varphi')^{-1} |\partial_\alpha(g^{\alpha j}\varphi_j)|^2 |v|^2 \, dxdt \\
&\quad + 8\,\mathrm{Re} \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} (\varphi')^{-1} \partial_\alpha(g^{\alpha j}\varphi_j)\varphi_k g^{k\beta}\partial_\beta v \bar{v} \, dxdt.
\end{aligned}
$$

Here, $(\varphi')^{-1}$ denotes $1/\varphi'$ (as opposed to the inverse of φ'). We will use this notation throughout.

Proceeding similarly to the estimate on the previous term (using Young's inequality and asymptotic flatness), we get the lower bound

$$2\left\|(\varphi')^{-1/2}P^i_\varphi v\right\|^2_{L^2_tL^2_x} \gtrsim 8\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\partial_r v|^2\,dxdt + \frac{2}{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^{-1}}{r^4}(\varphi'+\varphi'')^2|v|^2\,dxdt$$
$$- 4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{|\partial v|^2}{r^2}\,dxdt - 4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{3(\varphi')^2-2\varphi'\varphi''+(\varphi'')^2}{r^4}|v|^2\,dxdt$$
$$- \|g-m\|_{AF_{>R_0}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\partial v|^2\,dxdt$$
$$- \|g-m\|_{AF_{>R_0}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^3}{r^4}|v|^2\,dxdt. \tag{4.5}$$

Notice that the $|v|^2$ terms on the first and second lines of (4.5) feature weights of lower order than $\varphi''(\varphi')^2$ and, thus, will be negligible for large enough λ in comparison to the highest-order $|v|^2$ terms in (4.4).

Finally, we consider the commutator term. First, we compute that

$$[P^r_\varphi, P^i_\varphi] = 2D_\alpha g^{\alpha\beta}[\partial_\beta(\varphi_j g^{j\gamma})]D_\gamma + 2D_\gamma g^{\alpha\beta}[\partial_\alpha(g^{\gamma j}\varphi_j)]D_\beta - 2D_\alpha\varphi_j g^{j\gamma}[\partial_\gamma g^{\alpha\beta}]D_\beta$$
$$+ ig^{\alpha\beta}[\partial_\alpha\partial_\gamma(g^{\gamma j}\varphi_j)]D_\beta - iD_\alpha g^{\alpha\beta}[\partial_\beta\partial_\gamma(g^{\gamma j}\varphi_j)] + 2\varphi_k g^{k\gamma}[\partial_\gamma(\varphi_i g^{ij}\varphi_j)].$$

Integrating by parts gives that

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}[P^r_\varphi, P^i_\varphi]v\bar{v}\,dxdt = 2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}g^{\alpha\beta}\partial_\beta(\varphi_j g^{j\gamma})\partial_\gamma v\overline{\partial_\alpha v}\,dxdt + 2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}g^{\alpha\beta}\partial_\alpha(g^{\gamma j}\varphi_j)\partial_\beta v\overline{\partial_\gamma v}\,dxdt$$
$$- 2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\varphi_j g^{j\gamma}\partial_\gamma g^{\alpha\beta}\partial_\beta v\overline{\partial_\alpha v}\,dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}g^{\alpha\beta}\partial_\alpha\partial_\gamma\left(g^{\gamma j}\varphi_j\right)\partial_\beta v\bar{v}\,dxdt$$
$$+ \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}g^{\alpha\beta}\partial_\beta\partial_\gamma\left(g^{\gamma j}\varphi_j\right)v\overline{\partial_\alpha v}\,dxdt + 2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\varphi_k g^{k\gamma}\left(\partial_\gamma\varphi_i g^{ij}\varphi_j\right)|v|^2\,dxdt.$$

Once again, we utilize Young's inequality for products and replace g by m to obtain the lower bound

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}[P_\varphi^r, P_\varphi^i]v\bar{v}\,dxdt \gtrsim 4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\varphi_{ij}\partial_i v\overline{\partial_j v}\,dxdt$$
$$-\sum_{j,k=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(r\varphi_{jjk})^2|v|^2\,dxdt - \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{|\partial v|^2}{r^2}\,dxdt + 4\sum_{j,k=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\varphi_j\varphi_{jk}\varphi_k|v|^2\,dxdt$$
$$-\|g-m\|_{AF_{>R_0}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\partial v|^2\,dxdt - \|g-m\|_{AF_{>R_0}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^3}{r^4}|v|^2\,dxdt.$$

Notice that

$$\sum_{j,k=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(r\varphi_{jjk})^2|v|^2\,dxdt \lesssim \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^2}{r^4}|v|^2\,dxdt,$$

whereas

$$\sum_{j,k=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\varphi_j\varphi_{jk}\varphi_k|v|^2\,dxdt = \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^2}{r^4}(\varphi''-\varphi')|v|^2\,dxdt.$$

In particular, the former term is lower-order than the latter term in λ. All together,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}[P_\varphi^r, P_\varphi^i]v\bar{v}\,dxdt &\gtrsim 4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\varphi_{ij}\partial_i v\overline{\partial_j v}\,dxdt - \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{|\partial v|^2}{r^2}\,dxdt\\ &+\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{4(\varphi')^2}{r^4}(\varphi''-\varphi')|v|^2\,dxdt\\ &-\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^2}{r^4}|v|^2\,dxdt\\ &-\|g-m\|_{AF_{>R_0}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\partial v|^2\,dxdt\\ &-\|g-m\|_{AF_{>R_0}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^3}{r^4}|v|^2\,dxdt.\end{aligned}\tag{4.6}$$

Putting (4.4)-(4.6) together and using the negligibility of the described lower-order terms for large enough λ yields the lower bound

$$
\begin{aligned}
(4.7)\qquad & \langle [P^r_\varphi, P^i_\varphi]v, v\rangle + 2\left\|(\varphi')^{-1/2}P^i_\varphi v\right\|^2_{L^2_tL^2_x} + \left\langle\left(\frac{\varphi''}{2r^2}-\frac{4\varphi'}{r^2}\right)v, P^r_\varphi v\right\rangle \\
&\gtrsim 4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\varphi_{ij}\partial_i v\overline{\partial_j v}\,dxdt + 8\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\partial_r v|^2\,dxdt \\
&\quad + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\left(4\varphi'-\frac{1}{2}\varphi''-\frac{11}{2}\right)}{r^2}|\partial_t v|^2\,dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\left(\frac{1}{2}\varphi''-4\varphi'-\frac{11}{2}\right)}{r^2}|\nabla_x v|^2\,dxdt \\
&\quad + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left(\frac{4(\varphi')^3}{r^4}-\frac{\varphi''(\varphi')^2}{2r^4}\right)|v|^2\,dxdt + 4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^2}{r^4}(\varphi''-\varphi')|v|^2\,dxdt \\
&\quad + \frac{2}{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^{-1}}{r^4}(\varphi'+\varphi'')^2|v|^2\,dxdt - 4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{3(\varphi')^2-2\varphi'\varphi''+(\varphi'')^2}{r^4}|v|^2\,dxdt \\
&\quad - \frac{1}{2}\sum_{j=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left(r\partial_j\left(\frac{\varphi''}{2r^2}-\frac{4\varphi'}{r^2}\right)\right)^2|v|^2\,dxdt - \sum_{j,k=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(r\varphi_{jjk})^2\,|v|^2\,dxdt \\
&\quad - \|g-m\|_{AF_{>R_0}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\partial v|^2\,dxdt - \|g-m\|_{AF_{>R_0}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^3}{r^4}|v|^2\,dxdt \\
&\gtrsim 4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\varphi_{ij}\partial_i v\overline{\partial_j v}\,dxdt + 8\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\partial_r v|^2\,dxdt \\
&\quad + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\left(4\varphi'-\frac{1}{2}\varphi''-\frac{11}{2}\right)}{r^2}|\partial_t v|^2\,dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\left(\frac{1}{2}\varphi''-4\varphi'-\frac{11}{2}\right)}{r^2}|\nabla_x v|^2\,dxdt \\
&\quad + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi''(\varphi')^2}{r^4}|v|^2\,dxdt - \|g-m\|_{AF_{>R_0}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\partial v|^2\,dxdt.
\end{aligned}
$$

Next, we calculate that

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\varphi_{ij}\partial_i v\overline{\partial_j v}\,dxdt &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{r^2\delta_{ij}\varphi' + x_ix_j\left(\varphi''-2\varphi'\right)}{r^4}\partial_i v\overline{\partial_j v}\,dxdt \\
&= \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\nabla_x v|^2\,dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi''-2\varphi'}{r^2}|\partial_r v|^2\,dxdt.
\end{aligned}
$$

Factoring this into (4.7) and utilizing both the smallness of $\|g-m\|_{>R_0}$ and the conditions on φ

finally give that

$$
\begin{aligned}
&\left\langle [P^r_\varphi, P^i_\varphi]v, v\right\rangle + 2\left\|(\varphi')^{-1/2}P^i_\varphi v\right\|^2_{L^2_tL^2_x} + \left\langle\left(\frac{\varphi''}{2r^2} - \frac{4\varphi'}{r^2}\right)v, P^r_\varphi v\right\rangle \\
&\qquad\gtrsim \int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3} \frac{\left(4\varphi' - \frac{1}{2}\varphi'' - \frac{11}{2}\right)}{r^2}|\partial_t v|^2\,dxdt + \int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3} \frac{\left(\frac{1}{2}\varphi'' - \frac{11}{2}\right)}{r^2}|\nabla_x v|^2\,dxdt \\
&\qquad\quad + 4\int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3}\frac{\varphi''}{r^2}|\partial_r v|^2\,dxdt + \int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3}\frac{\varphi''(\varphi')^2}{r^4}|v|^2\,dxdt \\
&\qquad\quad - \|g-m\|_{AF_{>R_0}}\int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\partial v|^2\,dxdt \\
&\qquad\gtrsim \int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\partial_t v|^2\,dxdt + 4\int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3}\frac{\varphi''}{r^2}|\partial_r v|^2\,dxdt \\
&\qquad\quad + \int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3}\frac{\varphi''}{r^2}|\nabla_x v|^2\,dxdt + \int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^3}\frac{\varphi''(\varphi')^2}{r^4}|v|^2\,dxdt \\
&\qquad\gtrsim \left\|r^{-1}(1+\varphi'')^{1/2}\left(r^{-1}(1+\varphi')v, \nabla_x v\right)\right\|^2_{L^2_tL^2_x} + \left\|r^{-1}(1+\varphi')^{1/2}\partial_t v\right\|^2_{L^2_tL^2_x},
\end{aligned}
$$

which is precisely (4.3). □

Due to the support of u in the above proposition, (4.1) applies in the exterior region, making it appear amenable to pairing with an exterior estimate. However, exterior estimates require a constant weight, which violates the convexity assumption in the previous proposition. By modifying the weight to be constant in the exterior region, we introduce a lower-order error in a transition region (which is where we bend the weight to be constant).

Proposition 4.2. *Let P be an asymptotically flat damped wave operator, and $R > R_0$. Suppose that φ in an increasing function satisfying the assumptions given in Proposition* 4.1 *for $s < \ln R$ and constant for $s > \ln(2R)$. Then, for all $u \in \mathcal{S}(\mathbb{R}^4)$ with $\operatorname{supp} u \subset \{r > R_0\}$, we have the estimate*

(4.8)

$$
\begin{aligned}
&\left\|r^{-1}(1+\varphi''_+)^{1/2}e^{\varphi}\left(r^{-1}(1+\varphi')u, \nabla u\right)\right\|_{L^2_tL^2_{<R}} + \left\|r^{-1}(1+\varphi')^{1/2}e^{\varphi}\partial_t u\right\|_{L^2_tL^2_{<R}} + R^{-1/2}\left\|e^{\varphi}u\right\|_{LE^1_{>R}} \\
&\qquad \lesssim \|e^{\varphi}Pu\|_{L^2_tL^2_{<R}} + R^{-1/2}\|e^{\varphi}Pu\|_{LE^*_{>R}} + R^{-2}\left\|(1+\varphi')^{3/2}e^{\varphi}u\right\|_{L^2_tL^2_R},
\end{aligned}
$$

where $\varphi''_+(s) = \max\{\varphi''(s), 0\}$.

See Appendix B in [5] for the construction of such a weight. The proof of this proposition is highly similar to the prior proof.

Proof. Through a similar conjugation argument to that given in the proof of Proposition 4.1, it suffices to prove

$$\begin{aligned}(4.9)\quad &\left\| r^{-1}(1+\varphi''_+)^{1/2}\left(r^{-1}(1+\varphi')v, \nabla v\right)\right\|_{L^2_t L^2_{<R}} + \left\| r^{-1}(1+\varphi')^{1/2}\partial_t v\right\|_{L^2_t L^2_{<R}} + R^{-1/2}\|v\|_{LE^1_{>R}} \\ &\lesssim \|P_\varphi v\|_{L^2_t L^2_{<R}} + R^{-1/2}\|P_\varphi v\|_{LE^*_{>R}} + R^{-2}\left\|(1+\varphi')^{3/2} v\right\|_{L^2_t L^2_R},\end{aligned}$$

where $v = e^{\varphi}u$. To prove this estimate, we will consider three overlapping regions, namely

1. $\boldsymbol{R_1 := \{r < R\}}$**.** Here, we have (4.2). Note that this immediately implies (4.9) in this region.

2. $\boldsymbol{R_2 := \{R/4 < r < 2R\}}$**.** This is the transition region. The weight φ can be constructed (see [5]) so that $\varphi' \gtrsim 1$ and $|\varphi''| < \varphi'/2$. It is allowable for φ'' to be negative in this region, but we have the lower bound $\varphi'' > -\varphi'/2$. We will discuss this region in more depth momentarily.

3. $\boldsymbol{R_3 := \{r > 3R/2\}}$**.** Since $R > R_0$, we can apply the exterior estimate from Proposition 2.5. By a shifting argument (such as in Remark 1.5), we may leverage that $v \in \mathcal{S}(\mathbb{R}^4)$ to remove the initial energy term. However, this estimate is unweighted in φ. When $r > 2R$, the weight is constant, so (4.9) follows immediately from the exterior estimate. When $3R/2 < r \leq 2R$, we may assume that $|\varphi'| + |\varphi''| \ll 1$ (once again, see [5]). Hence, it is easy to bound the left-hand side of (4.9) by (2.3). Due to the aforementioned conditions on φ, we can see that $P = P_\varphi + E$, where E is of lower order and contains coefficients which are asymptotically flat (and hence small in this region) and contain derivatives of φ. Thus, these errors may be bootstrapped into the left-hand side of (4.9), which implies (4.9) in this region.

   Alternatively, this case can be dealt with similarly to step (1) by studying the specific construction of the Carleman weight present in [5] (in particular, it has a cutoff built in); see the aforementioned work for more.

We will elaborate on the transition region R_2, then we will combine the analysis using cutoffs.

Similar to the proof of Proposition 4.1, we will prove that the left-hand side of (4.9) is bounded above by

$$C\left(\langle [P_\varphi^r, P_\varphi^i]v, v\rangle + 2\left\|(\varphi')^{-1/2}P_\varphi^i v\right\|_{L_t^2L_x^2}^2 - \left\langle \frac{\varphi'}{r^2}v, P_\varphi^r v\right\rangle\right)$$

for v supported in $\{R/4 < r < 2R\}$. We have already computed the first two terms of the above in (4.4) and (4.5), and the third term is very similar to the third term on the right-hand side of (4.3). In particular, we can bound

$$\begin{aligned}
-\left\langle \frac{\varphi'}{r^2}v, P_\varphi^r v\right\rangle \gtrsim & \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{\varphi'}{r^2}|\partial_t v|^2\,dxdt - \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\nabla_x v|^2\,dxdt \\
& - \frac{1}{2}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left(\frac{4(\varphi')^2 + (\varphi'')^2 - 4\varphi'\varphi''}{r^4}\right)|v|^2\,dxdt - \frac{1}{2}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{|\partial v|^2}{r^2}\,dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{4(\varphi')^3}{r^4}|v|^2\,dxdt \\
& - \|g-m\|_{AF_{>R_0}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r^2}|\partial v|^2\,dxdt - \|g-m\|_{AF_{>R_0}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^3}{r^4}|v|^2\,dxdt.
\end{aligned}$$

Combining this with the estimates (4.4)-(4.5) and proceeding as in Proposition 4.1 provides (4.9) in the region R_2. We note that the right-hand side of (4.9) allows for a lower-order error term, and this emanates from the $|v|^2$ terms (which may have non-ideal sign due to the bounds $\varphi'' > -\varphi'/2$ as a result of bending the weight to be constant near infinity).

Now, we paste the analysis together for each region. Let χ_1, χ_2 be smooth cutoffs which are supported within R_1 and R_2 and identically 1 on $R_1 \setminus R_2$ and $R_2 \setminus R_1$, respectively. The bound in Proposition 2.5 already has the cutoff built in, so we do not need to introduce another one. Since we know that (4.9) holds within R_1, R_2, and R_3 individually, we have that

$$\begin{aligned}
\left\|r^{-1}(1+\varphi_+'')^{1/2}\left(r^{-1}(1+\varphi')v, \nabla v\right)\right\|_{L_t^2L_{<R}^2} + \left\|r^{-1}(1+\varphi')^{1/2}\partial_t v\right\|_{L_t^2L_{<R}^2} + R^{-1/2}\|v\|_{LE_{>R}^1} \\
\lesssim \|P_\varphi v\|_{L_t^2L_{<R}^2} + R^{-1/2}\|P_\varphi v\|_{LE_{>R}^*} + R^{-2}\left\|(1+\varphi')^{3/2}v\right\|_{L_t^2L_R^2} \\
+ \sum_{j=1}^{2}\|[P_\varphi, \chi_j]v\|_{L_t^2L_{<R}^2} + R^{-1/2}\|[P_\varphi,\chi_2]v\|_{LE_{>R}^*}.
\end{aligned} \tag{4.10}$$

Since χ_2 restricts to a compact region, the integration weight present in the LE^* norm can be removed. It is sufficient to analyze the commutators $\|[P_\varphi, \chi_j]v\|_{L_t^2 L_x^2}$ for $j = 1, 2$. One can check directly that

$$\|[P_\varphi, \chi_j]v\|_{L_t^2 L_x^2} \lesssim R^{-1} \|e^\varphi \partial v\|_{L_t^2 L^2_{R/4<|\cdot|<R}} + R^{-2} \|v\|_{L_t^2 L^2_{R/4<|\cdot|<R}}, \qquad j = 1, 2.$$

The second term on the right is bounded by

$$R^{-2} \left\|(1+\varphi')^{3/2} v\right\|_{L_t^2 L^2_{R/4<|\cdot|<R}}, \tag{4.11}$$

which is admissible on the upper bound side of (4.10). The first term on the right is bounded by

$$R^{-1} \|\partial v\|_{L_t^2 L^2_{R/4<|\cdot|<R}} + R^{-2} \left\|\varphi' v\right\|_{L_t^2 L^2_{R/4<|\cdot|<R}}. \tag{4.12}$$

The first term in (4.12) absorbs into the left-hand side of (4.10) since $\varphi', \varphi'' \gtrsim \lambda$ over the above integration region, while the second term is bounded by the aforementioned admissible term (4.11). This establishes (4.9), which concludes the proof. □

4.3 An Interior Carleman Estimate

The next Carleman estimate applies within a compact set. The analogous result in [27] is Proposition 5.3. The damping plays little role here.

Proposition 4.3. *Let P be an asymptotically flat damped wave operator and φ be a radial weight possessing the properties*

$$\begin{gathered} \varphi'(0) = 0, \qquad \varphi'' \approx \lambda + \sigma\varphi', \qquad |\varphi'''| \lesssim \sigma^2 \varphi' \\ 0 \le \varphi'' - \frac{\varphi'}{r} \lesssim \sigma\varphi' \text{ for all } r \ge 0, \qquad \frac{\varphi'}{r} \approx \varphi'' \text{ for all } r \ll_\sigma 1, \end{gathered}$$

where $\lambda, \sigma \gg 1$. Suppose further that ∂_t is uniformly time-like. Then, for all $u \in \mathcal{S}(\mathbb{R}^4)$, we have the estimate

$$\begin{aligned} \left\|(\varphi'/r)^{1/2} e^\varphi \partial u\right\|_{L_t^2 L_x^2} + \left\|(\varphi'')^{1/2} \varphi' e^\varphi u\right\|_{L_t^2 L_x^2} + \left\|(\varphi'/r) e^\varphi u\right\|_{L_t^2 L_x^2} \\ \lesssim \|e^\varphi P u\|_{L_t^2 L_x^2} + \left\|(\varphi'/\langle r\rangle)^{1/2} e^\varphi \partial_t u\right\|_{L_t^2 L^2_{\gtrsim 1}}. \end{aligned} \tag{4.13}$$

Once again, see Appendix B in [5] for the Carleman weight function construction. We note that the conditions on the weight imply that it is increasing and that $\varphi'/r \gtrsim \lambda$. In fact, further inspection of the weight present in [5] yields that $\varphi'/r \approx \lambda$ for $r \ll 1$.

Proof. Since $a \in C_c^\infty(\mathbb{R}^3)$ and $\varphi'/r \gtrsim \lambda$, the damping term $iaD_t u$ present in Pu absorbs into the left-most term in (4.13). Hence, it suffices to consider $P = D_\alpha g^{\alpha\beta} D_\beta$. By a similar conjugation argument to those given previously, it is enough to prove the estimate

(4.14)
$$\left\|(\varphi'/r)^{1/2}\partial v\right\|_{L_t^2 L_x^2} + \left\|\varphi'(\varphi'')^{1/2} v\right\|_{L_t^2 L_x^2} + \|r^{-1}\varphi' v\|_{L_t^2 L_x^2} \lesssim \|P_\varphi v\|_{L_t^2 L_x^2} + \left\|(\varphi'/\langle r\rangle)^{1/2}\partial_t v\right\|_{L_t^2 L^2_{\gtrsim 1}},$$

where $v = e^{\varphi} u$. The proof, once again, is a positive commutator argument. Our expression for the commutator is the same, but our weight is now radial, as opposed to being parameterized by $\ln r$. In particular, $\varphi_j = \dfrac{x_j \varphi'}{r}$, as opposed to $\dfrac{x_j\varphi'}{r^2}$. We write that

(4.15)
$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} [P_\varphi^r, P_\varphi^i] v\bar{v}\, dxdt$$
$$= 2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha\beta}\partial_\beta\left(\frac{x_j\varphi'}{r} g^{j\gamma}\right)\partial_\gamma v\overline{\partial_\alpha v}\, dxdt + 2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha\beta}\partial_\alpha\left(g^{\gamma j}\frac{x_j\varphi'}{r}\right)\partial_\beta v\overline{\partial_\gamma v}\, dxdt$$
$$- 2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{x_j\varphi'}{r} g^{j\gamma}\partial_\gamma g^{\alpha\beta}\partial_\beta v\overline{\partial_\alpha v}\, dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha\beta}\partial_\alpha\partial_\gamma\left(g^{\gamma j}\frac{x_j\varphi'}{r}\right)\partial_\beta v\bar{v}\, dxdt$$
$$+ \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha\beta}\partial_\beta\partial_\gamma\left(g^{\gamma j}\frac{x_j\varphi'}{r}\right) v\overline{\partial_\alpha v}\, dxdt + 2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{x_k\varphi'}{r} g^{k\gamma}\left(\partial_\gamma\frac{x_i\varphi'}{r} g^{ij}\frac{x_j\varphi'}{r}\right)|v|^2\, dxdt.$$

We will study the commutator in two overlapping regions which cover $\mathbb{R}^3$; the bounds that we obtain can be pasted together via a partition of unity as in the proof of Proposition 4.2.

(1) $\mathbf{supp}\, \boldsymbol{v} \subset \boldsymbol{R_1 := \{r \ll 1\}}$**.** Note that smaller powers of r generate dominant terms in this region. We will start with the $\partial_\alpha v\overline{\partial_\beta v}$-type terms in (4.15), namely the terms

$$2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha\beta}\partial_\beta\left(\frac{x_j\varphi'}{r}g^{j\gamma}\right)\partial_\gamma v\overline{\partial_\alpha v}\,dxdt + 2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha\beta}\partial_\alpha\left(g^{\gamma j}\frac{x_j\varphi'}{r}\right)\partial_\beta v\overline{\partial_\gamma v}\,dxdt$$
$$-2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{x_j\varphi'}{r}g^{j\gamma}\partial_\gamma g^{\alpha\beta}\partial_\beta v\overline{\partial_\alpha v}\,dxdt.$$

Using the chain rule, the above becomes

(4.16)
$$\begin{aligned}
&4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{x_j\varphi'}{r}g^{\alpha\beta}\partial_\beta g^{j\gamma}\partial_\gamma v\overline{\partial_\alpha v}\,dxdt + 4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha k}\left(\frac{x_jx_k\varphi''}{r^2}+\frac{\delta_{jk}\varphi'}{r}-\frac{x_jx_k\varphi'}{r^3}\right)g^{j\gamma}\partial_\gamma v\overline{\partial_\alpha v}\,dxdt\\
&\quad -2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{x_j\varphi'}{r}g^{j\gamma}\partial_\gamma g^{\alpha\beta}\partial_\beta v\overline{\partial_\alpha v}\,dxdt\\
&= 4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha j}\frac{\varphi'}{r}g^{j\gamma}\partial_\gamma v\overline{\partial_\alpha v}\,dxdt + 4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left(\varphi''-\frac{\varphi'}{r}\right)\frac{x_j}{r}g^{j\gamma}\partial_\gamma v\overline{\frac{x_k}{r}g^{k\alpha}\partial_\alpha v}\,dxdt\\
&\quad +4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{x_j\varphi'}{r}g^{\alpha\beta}\partial_\beta g^{j\gamma}\partial_\gamma v\overline{\partial_\alpha v}\,dxdt - 2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{x_j\varphi'}{r}g^{j\gamma}\partial_\gamma g^{\alpha\beta}\partial_\beta v\overline{\partial_\alpha v}\,dxdt.
\end{aligned}$$

Notice that the first term on the right-hand side of (4.16) can be written as

(4.17)
$$4\sum_{j=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}|g^{\alpha j}\partial_\alpha v|^2\,dxdt.$$

Since $\varphi''-\varphi'/r\geq 0$ by the assumptions on the weight, the second term on the right-hand side of (4.16) is non-negative and may be dropped when we bound from below. Since (4.17) includes division by $r \ll 1$, we may absorb the last two terms on the right-hand side of (4.16) into (4.17) for small enough r, which implies that the right-hand side of (4.16) is bounded below by a multiple of (4.17). We will hold off on bounding this term momentarily.

Next, we consider the mixed terms in (4.15), namely

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha\beta}\partial_\alpha\partial_\gamma(g^{\gamma j}\frac{x_j\varphi'}{r})\partial_\beta v\bar{v}\,dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha\beta}\partial_\beta\partial_\gamma\left(g^{\gamma j}\frac{x_j\varphi'}{r}\right)v\overline{\partial_\alpha v}\,dxdt.$$

We calculate that the above is equal to

$$6\,\mathrm{Re}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\ell\beta}g^{jk}\left(\varphi''-\frac{\varphi'}{r}\right)\left(\frac{\delta_{jk}x_\ell}{r^2}-\frac{x_jx_kx_\ell}{r^4}\right)\partial_\beta v\bar{v}\,dxdt$$
$$+2\,\mathrm{Re}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\ell\beta}g^{jk}\frac{x_jx_kx_\ell}{r^3}\varphi'''\partial_\beta v\bar{v}\,dxdt$$
$$+2\,\mathrm{Re}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha\beta}(\partial_\alpha g^{jk})\frac{x_jx_k}{r^2}\left(\varphi''-\frac{\varphi'}{r}\right)\partial_\beta v\bar{v}\,dxdt+2\,\mathrm{Re}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha\beta}(\partial_\alpha g^{jk})\frac{\delta_{jk}\varphi'}{r}\partial_\beta v\bar{v}\,dxdt$$
$$+2\,\mathrm{Re}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{\alpha\beta}(\partial_{\alpha\gamma}g^{\gamma j})\frac{x_j\varphi'}{r}\partial_\beta v\bar{v}\,dxdt+2\,\mathrm{Re}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{k\beta}(\partial_\gamma g^{\gamma j})\frac{x_jx_k}{r^2}\left(\varphi''-\frac{\varphi'}{r}\right)\partial_\beta v\bar{v}\,dxdt$$
$$+2\,\mathrm{Re}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} g^{k\beta}(\partial_\gamma g^{\gamma j})\frac{\delta_{jk}\varphi'}{r}\partial_\beta v\bar{v}\,dxdt.$$

By using Young's inequality for products, the smallness of r, the boundedness of the metric, and the conditions on φ, we can readily bound this above by a multiple of

$$\sigma\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left(\frac{\varphi'}{r}\right)^{3/2}|v|^2\,dxdt+\sigma\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left(\frac{\varphi'}{r}\right)^{1/2}|\partial v|^2\,dxdt. \tag{4.18}$$

Finally, we analyze the $|v|^2$ terms in (4.15):

$$2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{x_k\varphi'}{r}g^{k\gamma}\left(\partial_\gamma\frac{x_i\varphi'}{r}g^{ij}\frac{x_j\varphi'}{r}\right)|v|^2\,dxdt \tag{4.19}$$
$$=2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(\varphi')^3\frac{x_ix_jx_k}{r^3}g^{k\gamma}\left(\partial_\gamma g^{ij}\right)|v|^2\,dxdt+4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(\varphi')^2\varphi''\frac{x_ix_jx_kx_\ell}{r^4}g^{k\ell}g^{ij}|v|^2\,dxdt$$
$$+4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^3}{r}\frac{x_jx_k}{r^2}g^{k\ell}g^{\ell j}|v|^2\,dxdt-4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^3}{r}\frac{x_ix_jx_kx_\ell}{r^4}g^{k\ell}g^{ij}|v|^2\,dxdt$$
$$=2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(\varphi')^3\frac{x_ix_jx_k}{r^3}g^{k\gamma}\left(\partial_\gamma g^{ij}\right)|v|^2\,dxdt+4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^3}{r}\frac{x_jx_k}{r^2}g^{j\ell}g^{\ell k}|v|^2\,dxdt$$
$$+4\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left((\varphi')^2\varphi''-\frac{(\varphi')^3}{r}\right)\left(g^{ij}\frac{x_ix_j}{r^2}\right)^2|v|^2\,dxdt$$
$$\gtrsim\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(\varphi')^2\varphi''|v|^2\,dxdt,$$

where we have used that $\varphi'' - \varphi'/r \geq 0$ to drop the last term on the left of the inequality, the smallness of r and the boundedness of the metric to bootstrap the first term into the second term, that

$$\frac{(\varphi')^3}{r}\frac{x_j x_k}{r^2} g^{j\ell} g^{\ell k}|v|^2 \gtrsim \frac{(\varphi')^3}{r}|v|^2 \gtrsim (\varphi')^2\varphi''|v|^2$$

due to the ellipticity of g^{ij}, and the properties of the weight (respectively).

Combining (4.17)-(4.19), we have shown that

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} [P_\varphi^r, P_\varphi^i] v\bar{v}\, dxdt \gtrsim \sum_{j=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{\varphi'}{r}|g^{\alpha j}\partial_\alpha v|^2\, dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} (\varphi')^2\varphi''|v|^2\, dxdt \\ - \sigma \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{\varphi'}{r}\right)^{3/2}|v|^2\, dxdt - \sigma \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{\varphi'}{r}\right)^{1/2}|\partial v|^2\, dxdt.\end{aligned}$$

Finally, we consider the correction term $-\left\langle \frac{\varphi'}{r}v, P_\varphi^r v\right\rangle$. Through a similar process to our prior bounds (along with the positive-definiteness of g^{ij} and uniformly time-like nature of ∂_t), we compute that

$$\begin{aligned}(4.20)\qquad -\left\langle \frac{\varphi'}{r}v, P_\varphi^r v\right\rangle &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} (\varphi')^2\frac{\varphi'}{r}\frac{x_i x_j}{r^2} g^{ij}|v|^2\, dxdt - \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{\varphi'}{r} g^{\alpha\beta}\partial_\alpha v\overline{\partial_\beta v}\, dxdt \\ &\quad - \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{x_j}{r^2}\left(\varphi'' - \frac{\varphi'}{r}\right) g^{j\beta} v\overline{D_\beta v}\, dxdt \\ &\gtrsim \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} (\varphi')^2\varphi''|v|^2\, dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{\varphi'}{r}|\partial_t v|^2\, dxdt - \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{\varphi'}{r}|\nabla_x v|^2\, dxdt \\ &\quad - \sum_{j=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{\varphi'}{r}|g^{\alpha j}\partial_\alpha v|^2\, dxdt - \sigma^2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{\varphi'}{r}|v|^2\, dxdt.\end{aligned}$$

Hence, we have

$$\begin{aligned}
(4.21)\quad & \langle [P^r_\varphi, P^i_\varphi]v, v\rangle - \delta \left\langle \frac{\varphi'}{r}v, P^r_\varphi v\right\rangle \\
&\gtrsim (1-\delta)\sum_{j=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}|g^{\alpha j}\partial_\alpha v|^2\,dxdt + (1+\delta)\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(\varphi')^2\varphi''|v|^2\,dxdt + \delta\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}|\partial_t v|^2\,dxdt \\
&\qquad - \sigma\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left(\frac{\varphi'}{r}\right)^{3/2}|v|^2\,dxdt - \delta\sigma^2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}|v|^2\,dxdt \\
&\qquad - \sigma\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left(\frac{\varphi'}{r}\right)^{1/2}|\partial v|^2\,dxdt - \delta\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}|\nabla_x v|^2\,dxdt.
\end{aligned}$$

Notice that, by Young's inequality,

$$\begin{aligned}
\sum_{j=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}|g^{\alpha j}\partial_\alpha v|^2\,dxdt &= \sum_{j=1}^{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}\left(|g^{ij}\partial_i v|^2 + |g^{0j}\partial_t v|^2 + 2\,\mathrm{Re}\, g^{ij}\partial_i v\overline{g^{j0}\partial_t v}\right)\,dxdt \\
&\gtrsim \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}|\nabla_x v|^2\,dxdt.
\end{aligned}$$

Combining this with (4.21) yields the bound

$$\begin{aligned}
(4.22)\quad \langle [P^r_\varphi, P^i_\varphi]v, v\rangle - \delta \left\langle \frac{\varphi'}{r}v, P^r_\varphi v\right\rangle &\gtrsim (1+\delta)\left\|\varphi'(\varphi'')^{1/2}v\right\|^2_{L^2_tL^2_x} + \delta\left\|(\varphi'/r)^{1/2}\partial v\right\|^2_{L^2_tL^2_x} \\
&\quad - \sigma\left\|(\varphi'/r)^{3/4}v\right\|^2_{L^2_tL^2_x} - \delta\sigma^2\left\|(\varphi'/r)^{1/2}v\right\|^2_{L^2_tL^2_x} - \sigma\left\|(\varphi'/r)^{1/4}\partial v\right\|^2_{L^2_tL^2_x}.
\end{aligned}$$

We claim that we can add the term $\|(\varphi'/r)v\|_{L^2_tL^2_x}$ to the lower-bound side for free. Indeed, we first note that

$$(4.23)\qquad \left\|(\varphi'/r)v\right\|_{L^2_tL^2_x} \le \left\|\varphi''v\right\|_{L^2_tL^2_x},$$

since $r \ll 1$.

Converting to spherical-polar coordinates gives us that

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(\varphi'')^2|v|^2\,dxdt = \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{S^2}(\varphi''(r))^2|v(t,r\omega)|^2r^2\,drd\sigma(\omega)dt \tag{4.24}$$

$$= \frac{1}{3}\int_{-\infty}^{\infty}\int_{S^2}\int_{0}^{\infty}(\varphi''(r))^2|v(t,r\omega)|^2\partial_r r^3\,drd\sigma(\omega)dt$$

$$= -\frac{2}{3}\int_{-\infty}^{\infty}\int_{S^2}\int_{0}^{\infty}r\varphi''(r)\varphi'''(r)|v(t,r\omega)|^2r^2\,drd\sigma(\omega)dt$$

$$-\frac{2}{3}\int_{-\infty}^{\infty}\int_{S^2}\int_{0}^{\infty}r(\varphi''(r))^2\big(\operatorname{Re}v(t,r\omega)\partial_r\operatorname{Re}v(t,r\omega)$$

$$+\operatorname{Im}v(t,r\omega)\partial_r\operatorname{Im}v(t,r\omega)\big)r^2\,drd\sigma(\omega)dt$$

$$= -\frac{2}{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}r\varphi''\varphi'''|v|^2\,dxdt$$

$$-\frac{2}{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}r(\varphi'')^2\left(\operatorname{Re}v\ \partial_r\operatorname{Re}v+\operatorname{Im}v\ \partial_r\operatorname{Im}v\right)\,dxdt.$$

Using the conditions on φ, we obtain that

$$-\frac{2}{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}r\varphi''\varphi'''|v|^2\,dxdt-\frac{2}{3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}r(\varphi'')^2\left(\operatorname{Re}v\ \partial_r\operatorname{Re}v+\operatorname{Im}v\ \partial_r\operatorname{Im}v\right)\,dxdt \tag{4.25}$$

$$\lesssim\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}r\varphi''|\varphi'''||v|^2\,dxdt+\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}r(\varphi'')^2|v||\partial_r v|\,dxdt$$

$$\lesssim\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}r\varphi''|\varphi'''||v|^2\,dxdt+\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\varphi'\varphi''|v||\partial_r v|\,dxdt$$

$$\lesssim\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}r\varphi''|\varphi'''||v|^2\,dxdt+\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}r(\varphi'')^2\varphi'|v|^2\,dxdt+\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}|\partial_r v|^2\,dxdt$$

$$\lesssim\sigma^2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\varphi''(\varphi')^2|v|^2\,dxdt+\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}|\partial v|^2\,dxdt.$$

Combining (4.23)-(4.25) gives that

$$\|(\varphi'/r)v\|_{L_t^2L_x^2}^2 \lesssim \sigma^2 \left\|\varphi'(\varphi'')^{1/2}v\right\|_{L_t^2L_x^2}^2 + \left\|(\varphi'/r)^{1/2}\partial v\right\|_{L_t^2L_x^2}^2,$$

which implies that, in particular,

$$\frac{\delta}{\sigma^2}\|(\varphi'/r)v\|_{L_t^2L_x^2}^2 \lesssim \delta \left\|\varphi'(\varphi'')^{1/2}v\right\|_{L_t^2L_x^2}^2 + \delta\left\|(\varphi'/r)^{1/2}\partial v\right\|_{L_t^2L_x^2}^2.$$

Thus, this term may be freely added to the the right-hand side of (4.22), providing that

(4.26)

$$\begin{aligned}\langle [P_\varphi^r, P_\varphi^i]v, v\rangle - \delta\left\langle \frac{\varphi'}{r}v, P_\varphi^r v\right\rangle &\gtrsim \left\|\varphi'(\varphi'')^{1/2}v\right\|_{L_t^2L_x^2}^2 + \delta\left\|(\varphi'/r)^{1/2}\partial v\right\|_{L_t^2L_x^2}^2 + \frac{\delta}{\sigma^2}\|(\varphi'/r)v\|_{L_t^2L_x^2}^2 \\ &\quad - \sigma\left\|(\varphi'/r)^{3/4}v\right\|_{L_t^2L_x^2}^2 - \delta\sigma^2\left\|(\varphi'/r)^{1/2}v\right\|_{L_t^2L_x^2}^2 - \sigma\left\|(\varphi'/r)^{1/4}\partial v\right\|_{L_t^2L_x^2}^2.\end{aligned}$$

Since $\varphi'/r \approx \lambda \gg 1$ for $r \ll 1$, terms of lower power represent non-dominant terms. By shrinking δ if necessary and choosing σ, λ sufficiently large, we obtain that

$$\langle [P_\varphi^r, P_\varphi^i]v, v\rangle - \delta\left\langle \frac{\varphi'}{r}v, P_\varphi^r v\right\rangle \gtrsim \left\|\varphi'(\varphi'')^{1/2}v\right\|_{L_t^2L_x^2}^2 + \left\|(\varphi'/r)^{1/2}\partial v\right\|_{L_t^2L_x^2}^2 + \|(\varphi'/r)v\|_{L_t^2L_x^2}^2. \tag{4.27}$$

In particular, we effectively need that

$$\begin{aligned}\delta\lambda^{1/2} - \frac{\sigma\lambda^{1/4}}{2} &> 0 \\ \frac{\delta\lambda}{\sigma^2} - 2\sigma\lambda^{3/4} - \delta\sigma^2\lambda^{1/2} &> 0.\end{aligned}$$

We get that the left-hand sides are greater than 1/2 for e.g. $\delta = 1/8,\ \sigma = 10^6,\ \lambda = 7\cdot 10^{76}$ (in general, smaller δ and larger σ necessitate rather large λ). Applying the Schwarz inequality and Young's inequality for products to the left-hand side of (4.27) as in the proof of Proposition 4.1 establishes (4.14) for $r \ll 1$.

(2) **supp $\boldsymbol{v} \subset \boldsymbol{R_2} := \{\boldsymbol{r} \gtrsim \boldsymbol{1}\}$.** The work here is similar to the R_1 case, but we will require modifications whenever we made an argument dependent on r being small (namely what dominant

terms look like and that $\varphi'/r \approx \varphi''$ when $r \ll_\sigma 1$). The work is highly similar, so we will only describe the required changes that must be made.

For $\partial_\alpha v \overline{\partial_\beta v}$ terms in (4.15), we settle for a simple bound from below, namely that the quantity is bounded below by

$$-\left\|(\varphi'/r)^{1/2}\partial v\right\|_{L_t^2 L_x^2}^2 \tag{4.28}$$

(such a bound is immediate).

For the mixed terms in (4.15), we use that

$$\varphi'' - \frac{\varphi'}{r} \lesssim \sigma\varphi'$$

for terms involving $\varphi'' - \varphi'/r$,

$$|\varphi'''| \lesssim \sigma^2\varphi'$$

for the term involving φ''', and

$$\varphi'' \approx \lambda + \sigma\varphi'$$

to obtain that the mixed terms in the commmutator are bounded below by

$$-C(\sigma)\left(\left\|\varphi' v\right\|_{L_t^2 L_x^2}^2 + \left\|\lambda^{1/2} v\right\|_{L_t^2 L_x^2}^2 + \left\|\lambda^{1/2}\partial v\right\|_{L_t^2 L_x^2}^2\right). \tag{4.29}$$

The bound on the $|v|^2$ terms in (4.15) is the same as in the R_1 case (namely, (4.19)), although we utilize a different property of the weight to achieve it. In particular, we note that

$$(\varphi')^2\varphi'' \approx \lambda(\varphi')^2 + \sigma(\varphi')^3,$$

and so

$$(\varphi')^2\varphi'' \gg (\varphi')^3 \gtrsim_\sigma \frac{(\varphi')^3}{r}.$$

This allows us to absorb all poorly-signed terms into

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(\varphi')^2\varphi''\left(g^{ij}\frac{x_ix_j}{r^2}\right)^2|v|^2\,dxdt \gtrsim_\sigma \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(\varphi')^2\varphi''|v|^2\,dxdt.$$

Armed with (4.28), (4.19), and (4.29), we have that

$$C(\sigma)\left(\left\|\varphi'(\varphi'')^{1/2}v\right\|^2_{L^2_tL^2_x} - \|\varphi' v\|^2_{L^2_tL^2_x} - \left\|\lambda^{1/2}v\right\|^2_{L^2_tL^2_x} - \left\|\lambda^{1/2}\partial v\right\|^2_{L^2_tL^2_x} - \left\|(\varphi'/r)^{1/2}\partial v\right\|^2_{L^2_tL^2_x}\right)$$
$$\lesssim \left\langle [P^r_\varphi, P^i_\varphi]v, v\right\rangle.$$

Since $\varphi'' \approx \lambda + \sigma\varphi'$ and $\varphi' \gtrsim r\lambda$, a sufficiently large choice of λ guarantees that we may absorb the poorly-signed $|v|^2$ terms, providing that

$$C(\sigma)\left(\left\|\varphi'(\varphi'')^{1/2}v\right\|^2_{L^2_tL^2_x} - \left\|(\varphi'/r)^{1/2}\partial v\right\|^2_{L^2_tL^2_x}\right) \lesssim \left\langle [P^r_\varphi, P^i_\varphi]v, v\right\rangle.$$

Next, we consider the correction term $\gamma\left\langle \frac{\varphi'}{r}v, P^r_\varphi v\right\rangle$, where $\gamma > 0$ is to-be-determined (and bears no relation to the scaling parameter in the high frequency work). Similar to the work in the R_1 region, we compute that

$$\begin{aligned}\left\langle \frac{\varphi'}{r}v, P^r_\varphi v\right\rangle &= -\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(\varphi')^2\frac{\varphi'}{r}\frac{x_ix_j}{r^2}g^{ij}|v|^2\,dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}g^{\alpha\beta}\partial_\alpha v\overline{\partial_\beta v}\,dxdt \\ &\quad + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{x_j}{r^2}\left(\varphi'' - \frac{\varphi'}{r}\right)g^{j\beta}v\overline{D_\beta v}\,dxdt \\ &\gtrsim -\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{(\varphi')^3}{r}|v|^2\,dxdt - \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}|\partial_t v|^2\,dxdt + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}|\nabla_x v|^2\,dxdt \\ &\quad - \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\frac{\varphi'}{r}|v|^2\,dxdt,\end{aligned}$$

using an application of Young's inequality for products to absorb the missing term from (4.20). Combining this with the prior computation, we obtain the inequality

$$C(\sigma)\left\|\varphi'(\varphi'')^{1/2}v\right\|^2_{L^2_tL^2_x}+(\gamma-C(\sigma))\left\|(\varphi'/r)^{1/2}\partial v\right\|^2_{L^2_tL^2_x}\lesssim\langle[P^r_\varphi,P^i_\varphi]v,v\rangle$$
$$+\gamma\left(\left\langle\frac{\varphi'}{r}v,P^r_\varphi v\right\rangle+\left\|(\varphi'/r)^{1/2}\partial_t v\right\|^2_{L^2_tL^2_x}+\left\|((\varphi')^3/r)^{1/2}v\right\|^2_{L^2_tL^2_x}+\left\|(\varphi'/r)^{1/2}v\right\|^2_{L^2_tL^2_x}\right).$$

For sufficiently large γ and λ, the conditions on φ ensure that

$$\gamma>C(\sigma),\qquad \gamma\varphi'/r+\gamma(\varphi')^3/r<C(\sigma)(\varphi')^2\varphi''.$$

Thus,

$$\left\|\varphi'(\varphi'')^{1/2}v\right\|^2_{L^2_tL^2_x}+\left\|(\varphi'/r)^{1/2}\partial v\right\|^2_{L^2_tL^2_x}\lesssim\langle[P^r_\varphi,P^i_\varphi]v,v\rangle+\gamma\left\langle\frac{\varphi'}{r}v,P^r_\varphi v\right\rangle+\gamma\left\|(\varphi'/r)^{1/2}\partial_t v\right\|^2_{L^2_tL^2_x}.$$

Applying the Schwarz inequality and Young's inequality for products to the left-hand side establishes (4.14). □

4.4 The Medium Frequency Estimate

Now, we state the main theorem of this chapter - our medium frequency estimate. The corresponding theorem in [27] is Theorem 5.4.

Theorem 4.4. *Let P be an asymptotically flat damped wave operator, and suppose that ∂_t be uniformly time-like. Then, for any $\delta>0$, there exists a bounded, non-decreasing radial weight $\varphi=\varphi(\ln(1+r))$ so that for all $u\in\mathcal{S}(\mathbb{R}^4)$, we have the bound*

$$\left\|(1+\varphi''_+)^{1/2}e^\varphi(\nabla u,\langle r\rangle^{-1}(1+\varphi')u\right\|_{LE}+\left\|(1+\varphi')^{1/2}e^\varphi\partial_t u\right\|_{LE}$$
$$\lesssim\|e^\varphi Pu\|_{LE^*}+\delta\left(\left\|(1+\varphi')^{1/2}e^\varphi u\right\|_{LE}+\left\|\langle r\rangle^{-1}(1+\varphi''_+)^{1/2}(1+\varphi')e^\varphi\partial_t u\right\|_{LE}\right). \tag{4.30}$$

Since δ can be chosen arbitrarily, it will allow for any interval of frequencies bounded away from both zero and infinity.

Remark 4.5. Just as we did for the high frequency estimate, it is important to emphasize *why* this is an appropriate estimate on the medium frequencies. To that end, suppose that u is supported at time frequencies τ such that $0<\tau_0\leq|\tau|\leq\tau_1$, where $\tau_0<\tau_1$. For compatibility with the other frequency regimes, we will want $\tau_0 \ll 1 \ll \tau_1$. Plancherel's theorem yields that

$$\delta \left\| (1+\varphi')^{1/2} e^{\varphi} u \right\|_{LE} \lesssim \frac{\delta}{\tau_0} \left\| (1+\varphi')^{1/2} e^{\varphi} \partial_t u \right\|_{LE},$$

while

$$\delta \left\| \langle r \rangle^{-1} (1+\varphi''_+)^{1/2} (1+\varphi') e^{\varphi} \partial_t u \right\|_{LE} \lesssim \delta \tau_1 \left\| \langle r \rangle^{-1} (1+\varphi''_+)^{1/2} (1+\varphi') e^{\varphi} u \right\|_{LE}.$$

By choosing δ sufficiently small, both terms absorb into the left-hand side of (4.30) in a direct fashion. We can translate our work immediately into a local energy decay estimate for u, with an implicit constant which depends on φ. ■

Proof. We will apply Propositions 4.2 and 4.3 using cutoffs. However, we need to leave enough room for overlap. Let χ_1 be a smooth cutoff which is identically one for $\{|x| \leq 2R\}$ and supported in $\{|x| \leq 4R\}$, and let χ_2 be a smooth cutoff which is identically one for $\{|x| \geq 2R\}$ and supported in $\{|x| \geq R\}$. Then, we split u into pieces $u_{in} = \chi_1 u$ and $u_{out} = \chi_2 u$. Here, $R > R_0$.

Since Proposition 4.2 is meant to be applied in the exterior $\{|x| \gtrsim R\}$ and Proposition 4.3 in the interior $\{|x| \lesssim R\}$, we will take a weight φ which is consistent with the former weight in the interior and with the latter weight in the exterior. Since both weights feature parameters, we must choose them consistently. In particular, we first take σ sufficiently large, then we choose λ sufficiently large so that it works for both weights. Away from zero, the coordinates being radial or log radial does not matter so much, and we choose φ such that

$$\varphi'(s) \approx \min\{\lambda r^{-1}, \lambda \ln(1+r)\}, \qquad \varphi''(s) \approx \lambda, \qquad 1 \lesssim s \lesssim \ln R.$$

Notice that such a choice is consistent with the weight conditions present in the prior propositions.

Now, we will apply Proposition 4.3 to u_{in} and Proposition 4.2 to u_{out}. Evidently, we must analyze the commutators $[P, \chi_1]$ and $[P, \chi_2]$, which are of the form

$$[P, \chi_1] = \mathcal{O}(R^{-1} \mathbb{1}_{(2R,4R)}) \nabla + \mathcal{O}(R^{-2} \mathbb{1}_{(2R,4R)})$$
$$[P, \chi_2] = \mathcal{O}(R^{-1} \mathbb{1}_{(R,2R)}) \nabla + \mathcal{O}(R^{-2} \mathbb{1}_{(R,2R)}).$$

Applying the aforementioned propositions and the above estimate now give that

$$
\begin{aligned}
(4.31)\qquad &\left\|(\varphi'/r)^{1/2}e^{\varphi}\partial u_{in}\right\|_{L^2_tL^2_x} + \left\|(\varphi'')^{1/2}\varphi' e^{\varphi}u_{in}\right\|_{L^2_tL^2_x} + \left\|(\varphi'/r)e^{\varphi}u_{in}\right\|_{L^2_tL^2_x} \\
&\lesssim \left\|e^{\varphi}Pu\right\|_{L^2_tL^2_{\lesssim R}} + \left\|e^{\varphi}[P,\chi_1]u\right\|_{L^2_tL^2_x} + \left\|(\varphi'/\langle r\rangle)^{1/2}e^{\varphi}\partial_t u_{in}\right\|_{L^2_tL^2_{\gtrsim 1}} \\
&\lesssim \left\|e^{\varphi}Pu\right\|_{L^2_tL^2_{\lesssim R}} + R^{-1}\left\|e^{\varphi}\nabla u\right\|_{L^2_tL^2_R} \\
&\qquad + R^{-2}\left\|e^{\varphi}u\right\|_{L^2_tL^2_R} + \left\|(\varphi'/\langle r\rangle)^{1/2}e^{\varphi}\partial_t u_{in}\right\|_{L^2_tL^2_{\gtrsim 1}},
\end{aligned}
$$

and

$$
\begin{aligned}
(4.32)\qquad &\left\|r^{-1}(1+\varphi''_+)^{1/2}e^{\varphi}\left(r^{-1}(1+\varphi')u_{out},\nabla u_{out}\right)\right\|_{L^2_tL^2_{\lesssim R}} \\
&\quad + \left\|r^{-1}(1+\varphi')^{1/2}e^{\varphi}\partial_t u_{out}\right\|_{L^2_tL^2_{\lesssim R}} + R^{-1/2}\left\|e^{\varphi}u_{out}\right\|_{LE^1_{\gtrsim R}} \\
&\lesssim \left\|e^{\varphi}Pu\right\|_{L^2_tL^2_{\lesssim R}} + R^{-1/2}\left\|e^{\varphi}Pu\right\|_{LE^*_{\gtrsim R}} \\
&\quad + \left\|e^{\varphi}[P,\chi_2]u\right\|_{L^2_tL^2_{\lesssim R}} + R^{-1/2}\left\|e^{\varphi}[P,\chi_2]u\right\|_{LE^*_{\gtrsim R}} + R^{-2}\left\|(1+\varphi')^{3/2}e^{\varphi}u_{out}\right\|_{L^2_tL^2_R} \\
&\lesssim \left\|e^{\varphi}Pu\right\|_{L^2_tL^2_{\lesssim R}} + R^{-1/2}\left\|e^{\varphi}Pu\right\|_{LE^*_{\gtrsim R}} + R^{-1}\left\|e^{\varphi}\nabla u\right\|_{L^2_tL^2_R} + R^{-2}\left\|e^{\varphi}u\right\|_{L^2_tL^2_R} \\
&\quad + R^{-3/2}\left\|e^{\varphi}\nabla u\right\|_{LE^*_R} + R^{-5/2}\left\|e^{\varphi}u\right\|_{LE^*_R} + R^{-2}\left\|(1+\varphi')^{3/2}e^{\varphi}u_{out}\right\|_{L^2_tL^2_R}.
\end{aligned}
$$

Multiplying through by $R^{1/2}$ in (4.32) yields that

(4.33)

$$
\begin{aligned}
&\left\|(1+\varphi''_+)^{1/2}e^{\varphi}\left(r^{-1}(1+\varphi')u_{out},\nabla u_{out}\right)\right\|_{LE_{\lesssim R}} + \left\|(1+\varphi')^{1/2}e^{\varphi}\partial_t u_{out}\right\|_{LE_{\lesssim R}} + \left\|e^{\varphi}u_{out}\right\|_{LE^1_{\gtrsim R}} \\
&\qquad \lesssim \left\|e^{\varphi}Pu\right\|_{LE^*_{\lesssim R}} + \left\|e^{\varphi}Pu\right\|_{LE^*_{\gtrsim R}} + \left\|e^{\varphi}u\right\|_{LE^1_R} + R^{-3/2}\left\|(1+\varphi')^{3/2}e^{\varphi}u_{out}\right\|_{L^2_tL^2_R}.
\end{aligned}
$$

For (4.31), we utilize the properties of the weight to obtain that

$$
\begin{aligned}
(4.34)\qquad &\left\|(1+\varphi''_+)^{1/2}e^{\varphi}\left(r^{-1}(1+\varphi')u_{in},\nabla u_{in}\right)\right\|_{LE_{\lesssim R}} + \left\|(\varphi'/r)^{1/2}e^{\varphi}\partial_t u_{in}\right\|_{L^2_tL^2_x} \\
&\qquad \lesssim \left\|e^{\varphi}Pu\right\|_{LE^*_{\lesssim R}} + R^{-1/2}\left\|e^{\varphi}u\right\|_{LE^1_R} + \left\|(\varphi'/\langle r\rangle)^{1/2}e^{\varphi}\partial_t u_{in}\right\|_{L^2_tL^2_{\gtrsim 1}}.
\end{aligned}
$$

For large enough R and λ, the middle term on the right absorbs into the left-hand sides of (4.33) and (4.34); it is crucial here that the supports of the cutoffs χ_1 and χ_2 have enough overlap so that we may bootstrap. Combining this with (4.33), performing a similar absorption, then using more of the assumptions on φ give us the estimate

$$\begin{aligned}\left\|(1+\varphi''_+)^{1/2}e^{\varphi}\left(r^{-1}(1+\varphi')u,\nabla u\right)\right\|_{LE_{\lesssim R}} + \left\|(1+\varphi')^{1/2}e^{\varphi}\partial_t u\right\|_{LE_{\lesssim R}} + \|e^{\varphi}u\|_{LE^1_{\gtrsim R}} \\ \lesssim \|e^{\varphi}Pu\|_{LE^*_{\lesssim R}} + \|e^{\varphi}Pu\|_{LE^*_{\gtrsim R}} + R^{-3/2}\left\|(1+\varphi')^{3/2}e^{\varphi}u\right\|_{L^2_tL^2_R} + \left\|(\varphi'/\langle r\rangle)^{1/2}e^{\varphi}\partial_t u\right\|_{L^2_tL^2_{1\lesssim\cdot\lesssim R}}.\end{aligned}$$

Since the weight is constant for $r \gtrsim R$, the above becomes

$$\begin{aligned}\left\|(1+\varphi''_+)^{1/2}e^{\varphi}\left(r^{-1}(1+\varphi')u,\nabla u\right)\right\|_{LE} + \left\|(1+\varphi')^{1/2}e^{\varphi}\partial_t u\right\|_{LE} \\ \lesssim \|e^{\varphi}Pu\|_{LE^*} + R^{-3/2}\left\|(1+\varphi')^{3/2}e^{\varphi}u\right\|_{L^2_tL^2_R} + \left\|(\varphi'/\langle r\rangle)^{1/2}e^{\varphi}\partial_t u\right\|_{L^2_tL^2_{1\lesssim\cdot\lesssim R}}.\end{aligned}$$

Now, we observe that

$$R^{-3/2}\left\|(1+\varphi')^{3/2}e^{\varphi}u\right\|_{L^2_tL^2_R} \approx R^{-1}|\varphi'(\ln R)|\left\|(1+\varphi')^{1/2}e^{\varphi}u\right\|_{L^2_tL^2_R}.$$

Since $R^{-1}\varphi'(\ln R) \to 0$ as $R \to \infty$, we can choose R large enough so that $|R^{-1}\varphi'(\ln R)| < \delta$. Similarly,

$$\begin{aligned}\left\|(\varphi'/\langle r\rangle)^{1/2}e^{\varphi}\partial_t u\right\|_{L^2_tL^2_{1\lesssim\cdot\lesssim R}} &\approx \left\|\left((\varphi'/\langle r\rangle)(1+\varphi''_+)\right)^{-1/2}\langle r\rangle^{-1}(1+\varphi''_+)^{1/2}(1+\varphi')e^{\varphi}\partial_t u\right\|_{L^2_tL^2_{1\lesssim\cdot\lesssim R}} \\ &\approx \lambda^{-1}\left\|\langle r\rangle^{-1}(1+\varphi''_+)^{1/2}(1+\varphi')e^{\varphi}\partial_t u\right\|_{L^2_tL^2_{1\lesssim\cdot\lesssim R}}.\end{aligned}$$

Choosing $\lambda > 1/\delta$ completes the proof. □

CHAPTER 5

Low Frequency Analysis

5.1 Introduction

In this chapter, we will establish a key local energy estimate in the low frequency regime. Let

$$P_0 = P\big|_{D_t=0} = D_i g^{ij}(t,x) D_j.$$

This represents P at time frequency zero, and we will utilize it to obtain information in a neighborhood of this frequency. Since ∂_t is uniformly time-like, P_0 is also uniformly elliptic. We will establish weighted elliptic estimates for the flat Laplacian Δ in order to get similar estimates for P_0. The operator P_0 is a special case of that found in [27], so all of the results in their work apply with almost no modification. We include the details here, following their steps throughout.

At low frequencies, *the* obstruction to local energy decay arises when P has a *resonance* at frequency zero.

Definition 5.1. A function u is called a *zero resonant state* for P if $u \in \mathcal{LE}_0$ is non-zero and $P_0 u = 0$. If, in addition, $u \in L^2$, then we call u a *zero eigenfunction.*

Recall that the definition of the $\mathcal{LE}$ space was provided in Section 3.7 (and the zero subscript notation was described in Section 1.2).

For a general wave operator P, such resonant states are annihilated by P while having finite energy. However, they also possess an infinite LE^1 norm when integrating in t over $[0,\infty)$, which violates local energy decay. Such states are ruled out in our context due to the uniform ellipticity of P_0.

A quantitative condition on the existence of such resonant states is as follows.

Definition 5.2. P is said to satisfy a *zero resolvent bound/zero non-resonance condition* if there exists some K_0, independent of t, such that

$$\|u\|_{\dot{H}^1} \leq K_0 \|P_0 u\|_{\dot{H}^{-1}} \qquad \forall u \in \dot{H}^1. \tag{5.1}$$

Proposition 2.10 of [27] demonstrates that a stationary wave operator P has no zero resonant states/zero eigenfunctions if and only if the zero non-resonance condition holds. In our problem, this condition is satisfied due to the uniform ellipticity of P_0.

5.2 Weighted Estimates for the Flat Laplacian

To start, we will require numerous weighted estimates pertaining to the flat Laplacian, which can also be found in [27] (Lemma 6.4).

Lemma 5.3. *The inverse of the Euclidean Laplacian, Δ^{-1}, satisfies the following estimates for $u \in \mathcal{S}(\mathbb{R}^3)$*

$$\left\| \langle x \rangle^{-1} u \right\|_{\mathcal{LE}} \lesssim \|\Delta u\|_{\mathcal{LE}^*}, \tag{5.2}$$

$$\left\| \langle x \rangle^{-2+s} \nabla^s u \right\|_{\mathcal{LE}^*} \lesssim \|\Delta u\|_{\mathcal{LE}^*}, \qquad s = 1, 2 \tag{5.3}$$

$$\|\langle x \rangle^s \nabla^s u\|_{\mathcal{LE}} \lesssim \|\langle x \rangle \Delta u\|_{\mathcal{LE}^*}, \qquad s = 0, 1, 2. \tag{5.4}$$

Here,

$$\nabla^s = \sum_{|\alpha|=s} \partial^\alpha.$$

Proof. We will demonstrate the proof only on (5.2) and (5.3) with $s = 1$, as all other cases are similar. Let $\Delta u = f$. By utilizing an appropriate partition of unity, one can write $f = \sum_{j=0}^{\infty} f_j$ with $\operatorname{supp} f_j \subset \{\langle x \rangle \approx 2^j\}$, in which case we can also decompose $u = \sum_{j=0}^{\infty} u_j$, with $\Delta u_j = f_j$. For each such j, the fundamental solution to Laplace's equation possesses an explicit expression

$$u_j(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^n} \frac{f_j(y)}{|x-y|}\, dy = -\frac{1}{4\pi} \int_{\langle y \rangle \approx 2^j} \frac{f_j(y)}{|x-y|}\, dy.$$

We will start with (5.2). Given any fixed $k \in \mathbb{Z}_{\geq 0}$, we apply the triangle inequality to write

$$\begin{aligned}\left\| \langle x \rangle^{-3/2} u \right\|_{L^2(\langle x \rangle \approx 2^k)} &\lesssim \sum_{j=0}^{\infty} \left\| \langle x \rangle^{-3/2} u_j \right\|_{L^2(\langle x \rangle \approx 2^k)} \\ &= \sum_{j \ll k}^{\infty} \left\| \langle x \rangle^{-3/2} u_j \right\|_{L^2(\langle x \rangle \approx 2^k)} \\ &\quad + \sum_{j \gg k}^{\infty} \left\| \langle x \rangle^{-3/2} u_j \right\|_{L^2(\langle x \rangle \approx 2^k)} \\ &\quad + \sum_{j \approx k}^{\infty} \left\| \langle x \rangle^{-3/2} u_j \right\|_{L^2(\langle x \rangle \approx 2^k)}.\end{aligned}$$

Splitting the sum in this manner will allow us to compare the weights in the norm with those arising from the fundamental solution.

If $j \ll k$, then

$$\begin{aligned}\left\| \langle x \rangle^{-3/2} u_j \right\|^2_{L^2(\langle x \rangle \approx 2^k)} &\approx \int\limits_{\langle x \rangle \approx 2^k} \langle x \rangle^{-3} \left(\int\limits_{\langle y \rangle \approx 2^j} \frac{f_j(y)}{|x-y|}\, dy \right)^2 dx \\ &\lesssim \int\limits_{\langle x \rangle \approx 2^k} (2^k)^{-3} (2^j)^3 \int\limits_{\langle y \rangle \approx 2^j} \frac{|f_j(y)|^2}{|x-y|^2}\, dy dx \\ &\approx \int\limits_{\langle x \rangle \approx 2^k} (2^k)^{-5} (2^j)^3 \int\limits_{\langle y \rangle \approx 2^j} \frac{|f_j(y)|^2}{\left|1 - \frac{y}{\langle x \rangle}\right|^2}\, dy dx \\ &\approx \int\limits_{\langle x \rangle \approx 2^k} (2^k)^{-5} (2^j)^3 \int\limits_{\langle y \rangle \approx 2^j} |f_j(y)|^2\, dy dx \\ &\lesssim (2^j/2^k)^2 \left\| \langle x \rangle^{1/2} f_j \right\|^2_{L^2(\langle x \rangle \approx 2^j)} \\ &\leq \left\| \langle x \rangle^{1/2} f_j \right\|^2_{L^2(\langle x \rangle \approx 2^j)}.\end{aligned}$$

The case when $j \gg k$ is similar.

When $j \approx k$, we first write

$$\left\| \langle x \rangle^{-3/2} u_j \right\|_{L^2(\langle x \rangle \approx 2^k)} = \left\| \chi_{2^k}(|x|) \langle x \rangle^{-3/2} u_j \right\|_{L^2}.$$

Since $j \approx k$, it follows that

$$\left|\chi_{2^k}(|x|)\langle x\rangle^{-3/2} u_j(x)\right| = \chi_{2^k}(|x|)\langle x\rangle^{-3/2}\left|\int\limits_{\langle y\rangle\approx 2^j} \frac{f_j(y)}{|x-y|}\,dy\right|$$
$$= \chi_{2^k}(|x|)\langle x\rangle^{-3/2}\left|\int\limits_{\mathbb{R}^n} \chi_{2^j}(|y|)\frac{f_j(y)}{|x-y|}\,dy\right|$$
$$\lesssim (2^k)^{-3/2}\int\limits_{\mathbb{R}^n} \chi_{\lesssim 2^j}(|x-y|)\frac{|f_j(y)|}{|x-y|}\,dy.$$

For further elaboration on the cut-off, if we say that, e.g. $k-1 \le j \le k+1$, then the triangle inequality allows us to conclude the product of the cut-offs is smooth and supported in $\{|x-y| \lesssim 2^j\}$, since

$$0 \le |x-y| \le 2^j + 2^k \lesssim 2^{j+1} \lesssim 2^j.$$

By Young's convolution inequality,

$$\left\|\langle x\rangle^{-3/2} u_j\right\|_{L^2(\langle x\rangle\approx 2^k)} \lesssim (2^k)^{-3/2}\left\||x|^{-1}\chi_{\lesssim 2^j}(|x|)\right\|_{L^1}\|f_j\|_{L^2}$$
$$\lesssim (2^k)^{-3/2}(2^j)^2\|f_j\|_{L^2} \approx \left\|\langle x\rangle^{1/2} f_j\right\|_{L^2(\langle x\rangle\approx 2^j)}.$$

Summarizing our work, we have shown that

$$\left\|\langle x\rangle^{-3/2} u_j\right\|_{L^2(\langle x\rangle\approx 2^k)} \lesssim \sum_{j=0}^{\infty}\left\|\langle x\rangle^{1/2} f_j\right\|_{L^2(\langle x\rangle\approx 2^j)} \lesssim \|f\|_{\mathcal{LE}^*},$$

independently of k. This concludes the proof of (5.2).

Now, we will establish (5.3) when $s = 1$. There are two primary differences between this case and the prior case. The first is that both sides of the desired inequality feature an $\mathcal{LE}^*$ norm, so we now require summability on the left-hand side. The second is that we must take derivatives of u, which will be handled using the increased decay rate of derivatives of the fundamental solution (taking s derivatives yields a decay rate of $|x|^{-1-s}$).

Given our expression for u_j, we can calculate that

$$|\nabla u_j(x)| \lesssim \int\limits_{\langle y\rangle \approx 2^j} \frac{|f_j(y)|}{|x-y|^2}\, dy.$$

We will estimate

$$\begin{aligned}
\left\| \langle x\rangle^{-1} \nabla u \right\|_{\mathcal{LE}^*} &\lesssim \sum_{j,k=0}^{\infty} \left\| \langle x\rangle^{-1/2} \nabla u_j \right\|_{L^2(\langle x\rangle \approx 2^k)} \\
&\lesssim \sum_{j=0}^{\infty} \sum_{k \gg j} \left\| \langle x\rangle^{-1/2} \nabla u_j \right\|_{L^2(\langle x\rangle \approx 2^k)} + \sum_{j=0}^{\infty} \sum_{j \approx k} \left\| \langle x\rangle^{-1/2} \nabla u_j \right\|_{L^2(\langle x\rangle \approx 2^k)} \\
&\quad + \sum_{j=0}^{\infty} \sum_{k \ll j} \left\| \langle x\rangle^{-1/2} \nabla u_j \right\|_{L^2(\langle x\rangle \approx 2^k)} .
\end{aligned}$$

When $j \approx k$, Young's convolution inequality now gives that

$$\left\| \langle x\rangle^{-1/2} \nabla u_j \right\|_{L^2(\langle x\rangle \approx 2^k)} \lesssim (2^k)^{-1/2} 2^j \left\| f_j \right\|_{L^2} \approx \left\| \langle x\rangle^{1/2} f_j \right\|_{L^2(\langle x\rangle \approx 2^j)} ,$$

and so

$$\sum_{j=0}^{\infty} \sum_{j \approx k} \left\| \langle x\rangle^{-1/2} \nabla u_j \right\|_{L^2(\langle x\rangle \approx 2^k)} \lesssim \sum_{j=0}^{\infty} \left\| \langle x\rangle^{1/2} f_j \right\|_{L^2(\langle x\rangle \approx 2^j)} \lesssim \|f\|_{\mathcal{LE}^*} .$$

When $k \gg j$, we similarly have that

$$\begin{aligned}
\left\| \langle x\rangle^{-1/2} \nabla u_j \right\|^2_{L^2(\langle x\rangle \approx 2^k)} &\approx \int\limits_{\langle x\rangle \approx 2^k} \langle x\rangle^{-1} \left(\int\limits_{\langle y\rangle \approx 2^j} \frac{f_j(y)}{|x-y|^2}\, dy \right)^2 dx \\
&\lesssim \int\limits_{\langle x\rangle \approx 2^k} (2^k)^{-1} (2^j)^3 \int\limits_{\langle y\rangle \approx 2^j} \frac{|f_j(y)|^2}{|x-y|^4}\, dydx \\
&\approx \int\limits_{\langle x\rangle \approx 2^k} (2^k)^{-5} (2^j)^3 \int\limits_{\langle y\rangle \approx 2^j} \frac{|f_j(y)|^2}{\left| 1 - \frac{y}{\langle x\rangle} \right|^4}\, dydx \\
&\approx \int\limits_{\langle x\rangle \approx 2^k} (2^k)^{-5} (2^j)^3 \int\limits_{\langle y\rangle \approx 2^j} |f_j(y)|^2\, dydx \\
&\lesssim (2^j/2^k)^2 \left\| \langle x\rangle^{1/2} f_j \right\|^2_{L^2(\langle x\rangle \approx 2^j)} ,
\end{aligned}$$

which yields that

$$\begin{aligned}
\sum_{j=0}^{\infty}\sum_{k\gg j}\left\|\langle x\rangle^{-1/2}\nabla u_j\right\|_{L^2(\langle x\rangle\approx 2^j)} &\lesssim \sum_{j=0}^{\infty}\sum_{k\gg j}2^{j-k}\left\|\langle x\rangle^{1/2}\nabla f_j\right\|_{L^2(\langle x\rangle\approx 2^j)}\\
&\approx \sum_{j=0}^{\infty}2^j 2^{-j}\left\|\langle x\rangle^{1/2}\nabla f_j\right\|_{L^2(\langle x\rangle\approx 2^j)}\\
&\lesssim \|f\|_{\mathcal{LE}^*}.
\end{aligned}$$

The case of $k \ll j$ is analogous, and putting these estimates together directly yields (5.3) when $s = 1$. The proofs of the remaining cases are highly similar to those already provided. □

5.3 Perturbative Estimates

Next, we establish various stationary local energy estimates related to the zero non-resonance condition. These are rooted in perturbations of the weighted estimates for Δ given by Lemma 5.3. Once again, we follow [27] (in particular, Lemma 6.5).

Proposition 5.4. *Suppose that P_0 is asymptotically flat and satisfies the zero non-resonance condition (5.1). Then, we have the following local energy estimates for $u \in \mathcal{S}(\mathbb{R}^3)$:*

(a)

$$\|\langle x\rangle u\|_{\mathcal{LE}^1} \lesssim K_0 \|\langle x\rangle P_0 u\|_{\mathcal{LE}^*}, \tag{5.5}$$

(b)

$$K_0^{-1}\|\langle x\rangle u\|_{\mathcal{LE}^1_{<K_0}} + \|u\|_{\mathcal{LE}^1_{>K_0}} + \left\|\langle x\rangle^{-1}\nabla u\right\|_{\mathcal{LE}^*_{>K_0}} \lesssim \|P_0 u\|_{\mathcal{LE}^*}, \tag{5.6}$$

(c) *If $R_1 \gg \max\{R_0, K_0\}$, then*

(5.7)

$$K_0^{-1}\|\langle x\rangle u\|_{\mathcal{LE}^1_{<K_0}} + \|u\|_{\mathcal{LE}^1_{K_0<|\cdot|<R_1}} + \left\|\langle x\rangle^{-1}\nabla u\right\|_{\mathcal{LE}^*_{K_0<|\cdot|<R_1}} \lesssim \|P_0 u\|_{\mathcal{LE}^*_{<R_1}} + \left\|r^{-1}\nabla(ru)\right\|_{\mathcal{LE}_{R_1}}.$$

Proof. To prove (5.5)-(5.6), we will first prove the results in two special cases which will allow for a perturbative proof in the general case.

Case 1: If the operator is Δ, then (5.5)-(5.6) follow with $K_0 = 1$ via Lemma 5.3.

Case 2: Now, consider the case when $\tilde{P}_0$ is a small AF perturbation of Δ. Write $\tilde{P}_0 = D_i \tilde{g}^{ij} D_j$, with $\|\tilde{g} - I\|_{AF} = \delta \ll 1$ (here, $I_{ij} = \delta_{ij}$), and notice that

$$\begin{aligned}\Delta &= \tilde{P}_0 - D_i \tilde{h}^{ij} D_j, \\ &= \tilde{P}_0 - \tilde{h}^{ij} D_i D_j - (D_i \tilde{h}^{ij}) D_j,\end{aligned}$$

where $\tilde{h}^{ij} = \tilde{g}^{ij} - m^{ij}$. By (5.4) with $s = 1, 2$ and Hölder's inequality,

$$\begin{aligned}\|\langle x\rangle \Delta u\|_{\mathcal{LE}^*} &\leq \left\|\langle x\rangle \tilde{P}_0 u\right\|_{\mathcal{LE}^*} + \left\|\langle x\rangle \tilde{h}^{ij} D_i D_j u\right\|_{\mathcal{LE}^*} + \left\|\langle x\rangle (D_i \tilde{h}^{ij}) D_j u\right\|_{\mathcal{LE}^*} \\ &\lesssim \left\|\langle x\rangle \tilde{P}_0 u\right\|_{\mathcal{LE}^*} + \delta \sum_{i,j=1}^{3} \left\|\langle x\rangle^2 D_i D_j u\right\|_{\mathcal{LE}} + \delta \left\|\langle x\rangle \nabla_x u\right\|_{\mathcal{LE}} \\ &\lesssim \left\|\langle x\rangle \tilde{P}_0 u\right\|_{\mathcal{LE}^*} + \delta \left\|\langle x\rangle \Delta u\right\|_{\mathcal{LE}^*},\end{aligned}$$

with the right-most term being absorbable into the left-hand side. This establishes (5.5) for small AF perturbations of Δ, and (5.6) is similar.

Case 3: Now, let P_0 be a general asymptotically flat operator satisfying the zero non-resonance condition. Let $\tilde{P}_0$ be a small AF perturbation of Δ which agrees with P_0 for $|x| > R_0$, and consider the operator $\tilde{P}_0^{-1}$. This operator exists since it is a small perturbation of Δ, and Δ^{-1} exists (recall that the space of invertible linear operators is open). Next, consider $\tilde{P}_0^{-1} P_0 u$. Notice that we may apply (5.5) to this function using the operator $\tilde{P}_0$, providing that

$$\left\|\langle x\rangle \tilde{P}_0^{-1} P_0 u\right\|_{\mathcal{LE}^1} \lesssim \left\|\langle x\rangle P_0 u\right\|_{\mathcal{LE}^*}.$$

We will show that the same bound holds for the error in estimating u by $\tilde{P}_0^{-1} P_0 u$. First, denote $\tilde{u} := u - \tilde{P}_0^{-1} P_0 u$. Applying P_0 yields

$$P_0 \tilde{u} = (\tilde{P}_0 - P_0) \tilde{P}_0^{-1} P_0 u.$$

Observe that $|x| > R_0$, then $P_0 = \tilde{P}_0$, which shows that $\operatorname{supp} P_0 \tilde{u} \subset \{|x| \leq R_0\}$. Since this set is compact, the weights can be ignored within this region, and we may consider standard L^2 norms (the implicit constant will depend on R_0, which is permissible). Via the prior bounds, it will suffice to prove (5.5) with $\tilde{u}$ on the left-hand side.

A key intermediate step to show is that

$$\|P_0\tilde{u}\|_{\mathcal{LE}^*} \lesssim \|P_0 u\|_{\mathcal{LE}^*} . \tag{5.8}$$

To see this, we note that if $\tilde{P}_0 = D_i \tilde{g}^{ij} D_j$, then

$$(\tilde{P}_0 - P_0)v = (\tilde{g}^{ij} - g^{ij})D_i D_j v + \left(D_i(\tilde{g}^{ij} - g^{ij})\right) D_j v.$$

Using (5.3) with $s = 1, 2$ and that $\tilde{P}_0$ is a small AF perturbation of Δ (allowing us to apply the work in case 2), we obtain that

$$\begin{aligned}
\|P_0\tilde{u}\|_{\mathcal{LE}^*} &\lesssim \sum_{i,j=1}^{3} \left\| D_i D_j \tilde{P}_0^{-1} P_0 u \right\|_{\mathcal{LE}^*} + \left\| \langle x \rangle^{-1} \nabla \tilde{P}_0^{-1} P_0 u \right\|_{\mathcal{LE}^*} \\
&\lesssim \left\| \Delta \tilde{P}_0^{-1} P_0 u \right\|_{\mathcal{LE}^*} \\
&\lesssim \left\| \tilde{P}_0 \tilde{P}_0^{-1} P_0 u \right\|_{\mathcal{LE}^*} \\
&\lesssim \|P_0 u\|_{\mathcal{LE}^*} .
\end{aligned}$$

Now, we move on to establishing (5.5) for $\tilde{u}$. By utilizing (5.1), Plancherel's theorem, the Hardy inequality, the compact support of $P_0\tilde{u}$, and (5.8), we get that

$$\begin{aligned}
\|\tilde{u}\|_{\dot{H}^1_x} &\leq K_0 \|P_0\tilde{u}\|_{\dot{H}_x^{-1}} = K_0 \left\| |\xi|^{-1} \widehat{P_0\tilde{u}} \right\|_{L^2_\xi} \lesssim K_0 \left\| \nabla_\xi \widehat{P_0\tilde{u}} \right\|_{L^2_\xi} \lesssim K_0 \|\langle x \rangle P_0\tilde{u}\|_{L^2_x} \lesssim K_0 \|P_0\tilde{u}\|_{\mathcal{LE}^*} \\
&\lesssim K_0 \|P_0 u\|_{\mathcal{LE}^*} .
\end{aligned}$$

Combining the above with an application of the Hardy inequality on the L^2 piece (again, the weights do not matter when $|x| \leq 2R_0\}$) gives

$$\|\langle x \rangle \tilde{u}\|_{\mathcal{LE}^1_{\leq 2R_0}} \lesssim K_0 \left\| \langle x \rangle \tilde{P}_0 u \right\|_{\mathcal{LE}^*} .$$

For the exterior piece, write

$$\|\langle x\rangle \tilde{u}\|_{\mathcal{LE}^1_{>2R_0}} = \|\langle x\rangle \chi_{>2R_0}\tilde{u}\|_{\mathcal{LE}^1}.$$

Applying (5.5) using $\chi_{>2R_0}\tilde{u}$ and $\tilde{P}_0$ (which we may do since $\tilde{P}_0$ is a small AF perturbation) gives

$$\begin{aligned}\|\langle x\rangle \chi_{>2R_0}\tilde{u}\|_{\mathcal{LE}^1} &\lesssim \left\|\langle x\rangle \tilde{P}_0\chi_{>2R_0}\tilde{u}\right\|_{\mathcal{LE}^*}\\ &= \|\langle x\rangle P_0\chi_{>2R_0}\tilde{u}\|_{\mathcal{LE}^*}\\ &\lesssim \|\langle x\rangle P_0\tilde{u}\|_{\mathcal{LE}^*_{>2R_0}} + \|\langle x\rangle [P_0,\chi_{>2R_0}]\tilde{u}\|_{\mathcal{LE}^*}.\end{aligned}$$

We compute that the commutator is of the form

$$|[P_0,\chi_{>2R_0}]\tilde{u}| \lesssim R_0^{-1}\mathbb{1}_{[2R_0,4R_0]}(|g||\nabla\tilde{u}| + |\nabla g||\tilde{u}|) + R_0^{-2}\mathbb{1}_{[2R_0,4R_0]}|g||\tilde{u}|.$$

Since the terms are compactly-supported, the weights do not matter. Via another application of the Hardy inequality, we finally obtain that

$$\|\langle x\rangle \chi_{>2R_0}\tilde{u}\|_{\mathcal{LE}^1} \lesssim \|\langle x\rangle P_0\tilde{u}\|_{\mathcal{LE}^*_{>2R_0}} + \|\langle x\rangle [P_0,\chi_{>2R_0}]\tilde{u}\|_{\mathcal{LE}^*} \lesssim \|\langle x\rangle P_0\tilde{u}\|_{\mathcal{LE}^*_{>2R_0}} + \|\tilde{u}\|_{\dot{H}^1}.$$

The prior work for the $\dot{H}^1$ norm allows us to conclude the proof of (5.5). The argument for (5.6) is highly similar (and we have proven sufficient estimates on all of the relevant terms).

For (5.7), we define the function

$$\tilde{u}(x) = \chi_{<R_1}(|x|)u(x) + r^{-1}\chi_{>R_1}(|x|)(ru)_{R_1},$$

where $(ru)_{R_1}$ denotes the average of ru over $\{|x| \approx R_1\}$. Since $\Delta r^{-1} = 0$ away from $r = 0$, it follows that

$$P_0\tilde{u} = \chi_{<R_1}P_0u + [P_0,\chi_{R_1}](u - r^{-1}(ru)_{R_1}) + \chi_{>R_1}(ru)_{R_1}(P_0 - \Delta)r^{-1}.$$

Next, we apply (5.6) to obtain the estimate

$$
\begin{aligned}
(5.9)\qquad K_0^{-1}\left\|\langle x\rangle \tilde{u}\right\|_{\mathcal{LE}^1_{<K_0}} + \left\|\tilde{u}\right\|_{\mathcal{LE}^1_{>K_0}} + \left\|\langle x\rangle^{-1}\nabla\tilde{u}\right\|_{\mathcal{LE}^*_{>K_0}}& \\
\lesssim \left\|P_0\tilde{u}\right\|_{\mathcal{LE}^*}& \\
\lesssim \left\|P_0 u\right\|_{\mathcal{LE}^*_{\leq R_1}} + R_1^{-1}\left\|\nabla(u - r^{-1}(ru)_{R_1})\right\|_{\mathcal{LE}^*_{R_1}}& \\
+ R_1^{-2}\left\|u - r^{-1}(ru)_{R_1}\right\|_{\mathcal{LE}^*_{R_1}} + \left\|(ru)_{R_1}(P_0-\Delta)r^{-1}\right\|_{\mathcal{LE}^*_{>R_1}}&.
\end{aligned}
$$

We will first bound the right-hand side of (5.9). Using the Poincaré inequality, we directly obtain that

$$
\begin{aligned}
R_1^{-1}\left\|\nabla(u - r^{-1}(ru)_{R_1})\right\|_{\mathcal{LE}^*_{R_1}} + R_1^{-2}\left\|u - r^{-1}(ru)_{R_1}\right\|_{\mathcal{LE}^*_{R_1}}& \\
\lesssim R_1^{-1/2}\left\|\nabla(u - r^{-1}(ru)_{R_1})\right\|_{L^2_{R_1}} + R_1^{-3/2}\left\|u - r^{-1}(ru)_{R_1}\right\|_{L^2_{R_1}}& \\
\lesssim R_1^{-1/2}\left\|r^{-1}\nabla(ru)\right\|_{L^2_{R_1}} + R_1^{-3/2}\left\|u - r^{-1}(ru)_{R_1}\right\|_{L^2_{R_1}}& \\
\lesssim R_1^{-1/2}\left\|r^{-1}\nabla(ru)\right\|_{L^2_{R_1}}& \\
\lesssim \left\|\langle x\rangle^{-1}\nabla(ru)\right\|_{\mathcal{LE}_{R_1}}&.
\end{aligned}
$$

Next, we utilize asymptotic flatness to compute that

$$
\left\|(ru)_{R_1}(P_0-\Delta)r^{-1}\right\|_{\mathcal{LE}^*_{>R_1}} \lesssim \mathbf{c}R_1^{-1}|(ru)_{R_1}|.
$$

As for the left-hand side of (5.9), we check that

$$
K_0^{-1}\left\|\langle x\rangle \tilde{u}\right\|_{\mathcal{LE}^1_{<K_0}} = K_0^{-1}\left\|\langle x\rangle u\right\|_{\mathcal{LE}^1_{<K_0}},
$$

$$
\left\|\tilde{u}\right\|_{\mathcal{LE}^1_{>K_0}} \gtrsim \max\{\left\|u\right\|_{\mathcal{LE}^1_{K_0<|\cdot|<R_1}}, R_1^{-1}|(ru)_{R_1}|\},
$$

and

$$
\left\|\langle x\rangle^{-1}\nabla\tilde{u}\right\|_{\mathcal{LE}^*_{>K_0}} \approx \left\|\langle x\rangle^{-1}\nabla u\right\|_{\mathcal{LE}^*_{K_0<|\cdot|<R_1}} + R_1^{-1}|(ru)_{R_1}|.
$$

Putting everything together provides us with

$$K_0^{-1}\|\langle x\rangle u\|_{\mathcal{LE}^1_{<K_0}}+\|u\|_{\mathcal{LE}^1_{K_0<|\cdot|<R_1}}+\left\|\langle x\rangle^{-1}\nabla u\right\|_{\mathcal{LE}^*_{K_0<|\cdot|<R_1}}+R_1^{-1}|(ru)_{R_1}|$$
$$\lesssim\|P_0u\|_{\mathcal{LE}^*_{\leq R_1}}+\left\|\langle x\rangle^{-1}\nabla(ru)\right\|_{\mathcal{LE}_{R_1}}+\mathbf{c}R_1^{-1}|(ru)_{R_1}|.$$

Since $\mathbf{c}\ll 1$, the right-most term absorbs into the left, which we then drop to obtain (5.7). □

5.4 The Low Frequency Estimate

Finally, we have the main theorem of the chapter (compare to Theorem 6.1 in [27]). The damping plays an insignificant role here.

Theorem 5.5. *Let P be an asymptotically flat damped wave operator, and suppose that ∂_t is uniformly time-like. Then,*

$$\|u\|_{LE^1}\lesssim\|\partial_t u\|_{LE^1_c}+\|Pu\|_{LE^*}$$

for all $u\in\mathcal{S}(\mathbb{R}^4)$.

Once again, the implicit constant will depend on $\mathbf{c}$.

Remark 5.6. The error term $\|\partial_t u\|_{LE^1_c}$ has the unfortunate effect of requiring information on the size of first-order derivatives of $\partial_t u$. However, this estimate will only be used when the time frequency is close to zero, in which case this term will be absorbable into the left-hand side of the inequality. Indeed, if we consider $u\in\mathcal{S}(\mathbb{R}^4)$ with frequency support $0\leq|\tau|\leq\tau_0\ll 1$, then we may apply Plancherel's theorem to obtain that

$$\|\partial_t u\|_{LE^1_c}\lesssim\tau_0\|u\|_{LE^1_c}.$$

If τ_0 is sufficiently small, then we may absorb this term into the lower-bound side of (5.5) to obtain local energy decay for such u. ■

Proof. We will pair equation (5.7) in Lemma 5.4 with the exterior local energy estimate (2.2). Choose $R_1\gg R_0,K_0$ and let $R\in[K_0,R_1]$ (we will specify a choice shortly). Applying Proposition 2.4 with the given R yields

$$\|u\|_{LE^1_{>R}}\lesssim\|\partial u\|_{LE_R}+\|Pu\|_{LE^*_{>R}}.$$

Next, we take L^2 norms of (5.7) in time to obtain

$$K_0^{-1} \|\langle x\rangle u\|_{LE^1_{<K_0}} + \|u\|_{LE^1_{K_0<|\cdot|<R_1}} \lesssim \|P_0 u\|_{LE^*_{<R_1}} + \|u\|_{LE^1_{R_1}} .$$

Combining the above two estimates and noting that $\|u\|_{LE^1_{R_1}} \leq \|u\|_{LE^1_{>R}}$ give us

$$K_0^{-1} \|\langle x\rangle u\|_{LE^1_{<K_0}} + \|u\|_{LE^1_{>K_0}} \lesssim \|\partial_t u\|_{LE_R} + \|\nabla u\|_{LE_R} + \|Pu\|_{LE^*_{>R}} + \|P_0 u\|_{LE^*_{<R_1}} . \tag{5.10}$$

We must bound the gradient term, which will dictate the choice of R. Via (5.7), we have that

$$\sum_{\log_2 K_0 \leq j \leq \log_2 R_1} \left\|\langle x\rangle^{-1/2} \nabla u\right\|_{L^2_{2^j}} = \left\|\langle x\rangle^{-1} \nabla u\right\|_{\mathcal{LE}^*_{K_0<|\cdot|<R_1}} \lesssim \|P_0 u\|_{\mathcal{LE}^*_{<R_1}} + \|u\|_{\mathcal{LE}^1_{R_1}} .$$

We choose R such that

$$\|\nabla u\|_{\mathcal{LE}_R} \approx \left\|\langle x\rangle^{-1/2} \nabla u\right\|_{L^2_R} \leq \frac{1}{\log_2(R_1/K_0)} \sum_{\log_2 K_0 \leq j \leq \log_2 R_1} \left\|\langle x\rangle^{-1/2} \nabla u\right\|_{L^2_{2^j}} .$$

Combining the two previous inequalities together and taking L^2 norms in time give us that

$$\|\nabla u\|_{LE_R} \lesssim \|P_0 u\|_{LE^*_{<R_1}} + \frac{1}{\log_2(R_1/K_0)} \|u\|_{LE^1_{R_1}} .$$

Applying this to (5.10) yields that

$$K_0^{-1} \|\langle x\rangle u\|_{LE^1_{<K_0}} + \|u\|_{LE^1_{>K_0}} \lesssim \|\partial_t u\|_{LE_R} + \|Pu\|_{LE^*_{>R}} + \|P_0 u\|_{LE^*_{<R_1}} + \frac{1}{\log_2(R_1/K_0)} \|u\|_{LE^1_{R_1}} .$$

The last term on the right absorbs into the second term on the left for large enough R_1.

It remains to replace the P_0 on the right with P. This is straightforward, since

$$P_0 = P - D_t g^{00} D_t - D_t g^{0j} D_j - D_j g^{j0} D_t - iaD_t,$$

which implies that

$$\|P_0 u\|_{LE^*_{<R_1}} \lesssim_{R_1} \|Pu\|_{LE^*_{<R_1}} + \|\partial_t u\|_{LE^1_{<R_1}} .$$

□

CHAPTER 6

Establishing Local Energy Decay

In this section, we prove Theorem 1.8, local energy decay. In order to establish local energy decay for non-trapped waves, the authors in [27] first proved a version of local energy decay where one removes the Cauchy data at times 0 and T. This makes it significantly easier to perform frequency localization. We will prove this result here; the corresponding theorem in [27] is Theorem 7.1.

Theorem 6.1. *Let P stationary, asymptotically flat damped wave operator satisfying the geometric control condition (3.1), and suppose that ∂_t is uniformly time-like. Then, the estimate*

$$\|u\|_{LE^1} \lesssim \|Pu\|_{LE^*} \tag{6.1}$$

holds for all $u \in \mathcal{S}(\mathbb{R}^4)$.

This is where we will apply our high, medium, and low frequency analyses. For this reason, the implicit constant will depend on $\mathbf{c}$.

Proof. We will utilize a time-frequency partition of unity. In particular, we let $0 < \tau_0 \ll 1$ and $\tau_1 \gg 1$, so that Remarks 1.5, 4.5, and 5.6 apply (which hold as a consequence of Theorems 1.6, 4.4, and 5.5, respectively). Then, we can write

$$u = \chi_{|\tau|<\tau_0}(D_t)u + \chi_{\tau_0<|\tau|<\tau_1}(D_t)u + \chi_{|\tau|>\tau_1}(D_t)u =: \sum_{j=1}^{3} Q_j u.$$

From the aforementioned remarks, we directly obtain that

$$\|u\|_{LE^1} \lesssim \|Pu\|_{LE^*} + \sum_{j=1}^{3} \|[P, Q_j]u\|_{LE^*}.$$

Since the metric is stationary, the above commutators are zero, allowing us to conclude. □

If the metric is non-stationary and one has high, medium, and low frequency estimates in such a context, then the commutators in the above proof are non-zero. Provided that the operator is "almost-stationary" (see Definition 1.2 in [27]), the proof is more involved; see the proof of Theorem 7.1 in [27].

Armed with Theorem 6.1, we may proceed with a proof of Theorem 1.3. This is nearly identical to a proof given in [27] to establish a "two-point" local energy estimate (they did not necessarily possess a coercive energy nor a stationary wave operator).

Proof of Theorem 1.8. Let $Pu = f$. We first observe that we are guaranteed an intermediate estimate

$$\|\partial u\|_{L_t^\infty L_x^2[0,T]} \lesssim \varepsilon \|u\|_{LE^1[0,T]} + \|\partial u(0)\|_{L^2} + \varepsilon^{-1} \|f\|_{LE^*+L_t^1 L_x^2[0,T]}$$

for any $\varepsilon > 0$ by Corollary 2.3. Hence, we only need to show that

$$\|u\|_{LE^1[0,T]} \lesssim \|\partial u(0)\|_{L^2} + \|f\|_{LE^*+L_t^1 L_x^2[0,T]}, \tag{6.2}$$

in which case we may combine the above with the prior inequality and choose ε sufficiently small. In fact, we only need to demonstrate this when $\operatorname{supp} u \subset \{r \lesssim T\}$.

Indeed, we first write that

$$\|u\|_{LE^1[0,T]} \lesssim \|u\|_{LE^1_{r\lesssim T}[0,T]} + \|u\|_{LE^1_{r\gtrsim T}[0,T]}.$$

If $2^j = T$, then we first may take a supremum in time, apply the Hardy inequality, then apply Corollary 2.3 to get that

$$\begin{aligned}
\left\|\langle x\rangle^{-1} u\right\|_{LE_{r\gtrsim T}[0,T]} &= \sup_{k\gtrsim j} \left(\int_0^T \int_{A_k} \langle x\rangle^{-3} |u|^2 \, dxdt\right)^{1/2} \\
&\lesssim \sup_{k\gtrsim j} \left(\int_0^T \int_{A_k} T^{-1} |u/r|^2 \, dxdt\right)^{1/2} \\
&\lesssim \|u/r\|_{L_t^\infty L_x^2[0,T]} \lesssim \|\partial u\|_{L_t^\infty L_x^2[0,T]} \\
&\lesssim \|\partial u(0)\|_{L^2} + \varepsilon^{-1} \|f\|_{LE^*+L_t^1 L_x^2[0,T]} + \varepsilon \|u\|_{LE^1[0,T]},
\end{aligned}$$

where $\varepsilon > 0$ is arbitrary. The work for the derivative is similar (no need for the Hardy inequality here). Combining with the prior estimate, we get that

$$\|u\|_{LE^1[0,T]} \lesssim \|u\|_{LE^1_{r\lesssim T}[0,T]} + \|\partial u(0)\|_{L^2} + \varepsilon^{-1} \|f\|_{LE^*+L^1_t L^2_x[0,T]} + \varepsilon \|u\|_{LE^1[0,T]} .$$

By choosing ε sufficiently small, we can see that it suffices to prove (6.2) when $r \lesssim T$.

Suppose that v matches the Cauchy data of u at times 0 and T. After approximating u by Schwartz functions, $u - v$ satisfies the conditions of Theorem 6.1. Applying (6.1) to $u - v$ yields that

$$\|u\|_{LE^1[0,T]} \lesssim \|v\|_{LE^1} + \|u - v\|_{LE^1} \lesssim \|v\|_{LE^1} + \|P(u-v)\|_{LE^*} .$$

If v also satisfies that

$$\|v\|_{LE^1} + \|f - Pv\|_{LE^*} \lesssim \|\partial u(0)\|_{L^2} + \|f\|_{LE^*+L^1_t L^2_x[0,T]} ,$$

then the proof is complete. It remains to construct such a v. We will consider two overlapping regions in space, after which one can readily paste the constructions together via a partition of unity.

1. $\boldsymbol{R_1 := \{r < 4R_0\}}$. This region is dealt with exactly how we performed the case reductions (#2-3) to zero Cauchy data in Section 3.5: utilize a unit time interval partition of unity, restrict the forcing to such intervals, then match the initial (respectively final time) data on the first (respectively last) solution granted by the partition.

2. $\boldsymbol{R_2 := \{r > 2R_0\}}$. In this region, P is a small AF perturbation of $\Box$. Since we will construct a function localized here, no generality is lost in assuming that P is a small AF perturbation of $\Box$ everywhere. In particular, we may ignore the damping in this context. Further, we may assume that $u[T] = 0$ and $\operatorname{supp} f \subset \{t < 3T/4\}$ by a time-reversal argument (otherwise, we utilize a partition of unity in time and run through the argument below backwards in time with the Cauchy data being at time T).

   Let $\tilde{P}$ be a small AF perturbation of $\Box$ which equals P for $|x| > R_0$, and consider the

Cauchy problem

$$\tilde{P}w = f, \qquad w[0] = u[0].$$

Due to the assumptions on $\tilde{P}$, it satisfies local energy decay:

$$\|w\|_{LE^1[0,T]} + \|\partial w\|_{L^\infty_t L^2_x[0,T]} \lesssim \|\partial u(0)\|_{L^2} + \|f\|_{LE^*+L^1_t L^2_x[0,T]}\,. \tag{6.3}$$

Next, we truncate w to obtain the function

$$v(t,x) = \beta_{<T}(t)\chi_{>R_0}(|x|)w(t,x),$$

where $\beta_{<T}$ is smooth, supported in $[0,T]$, and identically one on the support of f. Applying the local energy decay estimate for $\tilde{P}$ to v instead and applying the Hardy inequality yields the estimate

$$\begin{aligned}\|v\|_{LE^1} &\lesssim \|\partial u(0)\|_{L^2} + \|Pv\|_{LE^*+L^1_t L^2_x} \\ &\lesssim \|\partial u(0)\|_{L^2} + \|f\|_{LE^*+L^1_t L^2_x[0,T]} + \|f - Pv\|_{LE^*}\,.\end{aligned}$$

It remains to establish an acceptable bound on $\|f - Pv\|_{LE^*[0,T]}$. Notice that

$$Pv(t,x) - f(t,x) = [P, \chi_{<R_0}(|x|)]\beta_{<T}(t)w(t,x) + \chi_{>R_0}(|x|)[P, \beta_{<T}(t)]w(t,x).$$

The first term is readily bounded above in LE^* by a multiple of $\|w\|_{LE^1[0,T]}$. For the second term, we observe that it is supported in $\Omega_T = \{(t,x) : 3T/4 \leq t < T,\ R_0 < |x| \lesssim T\}$. We immediately have the bound

$$\|[P, \beta_{<T}(t)]w\|_{LE^*} \lesssim T\,\|[P, \beta_{<T}(t)]w\|_{L^\infty_t L^2_x(\Omega_T)}\,.$$

By the Hardy inequality and (6.3), we can bound the right-hand side of the above as

$$\begin{aligned} T\left\|[P,\beta_{<T}(t)]w\right\|_{L_t^\infty L_x^2(\Omega_T)} &\lesssim T^{-1}\left\|w\right\|_{L_t^\infty L_x^2(\Omega_T)} + \left\|\partial w\right\|_{L_t^\infty L_x^2(\Omega_T)} \\ &\lesssim \left(\left\|w/r\right\|_{L_t^\infty L_x^2[3T/4,T]} + \left\|\partial w\right\|_{L_t^\infty L_x^2[3T/4,T]}\right) \\ &\lesssim \left\|\partial w\right\|_{L_t^\infty L_x^2} \\ &\lesssim \left\|\partial u(0)\right\|_{L^2} + \left\|f\right\|_{LE^*+L_t^1 L_x^2[0,T]}, \end{aligned}$$

which proves the desired inequality in this region.

□

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