What the Heck is a Distribution? A Survey For the Non-Analyst

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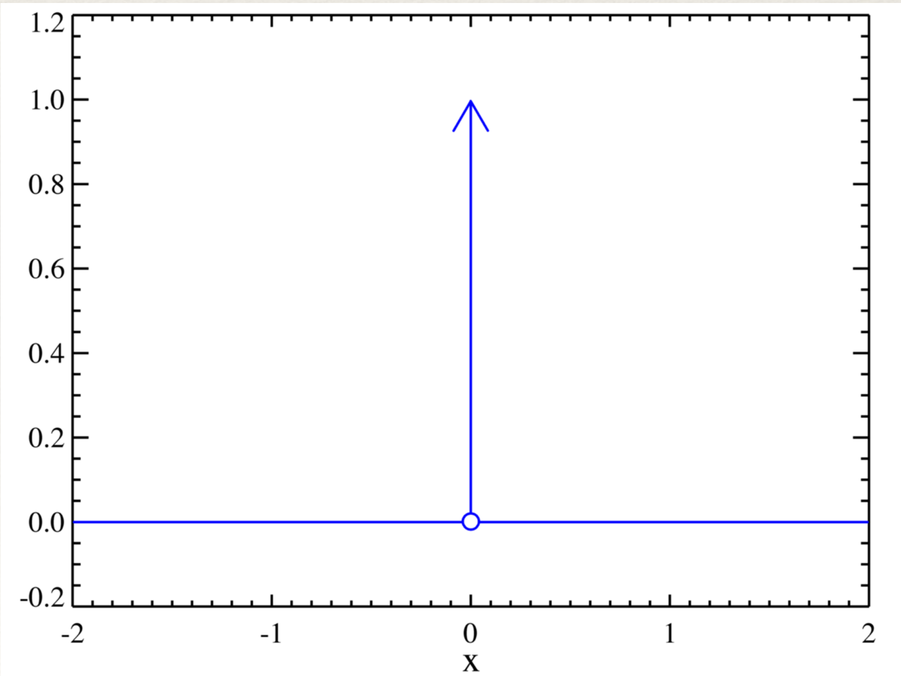
Historical Background on Distributions

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ +\infty & \text{if } x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

- Fourier→Dirac→Schwartz (1945)
- Goal was to make sense of things like the Dirac delta (meant to measure a point charge) i.e. mass distributions
- Also tie into things such as probability distributions, multi-pole expansions, etc.
- Will focus on tempered/Schwartz distributions



Schwartz Functions: $\mathcal{S}(\mathbb{R}^n)$

- Idea: Schwartz functions are smooth functions that decay faster than any polynomial
- Schwartz space:

$$\mathcal{S}(\mathbb{R}^n) := \left\{ \varphi \in C^\infty(\mathbb{R}^n) : \forall \alpha, \beta \in \mathbb{N}^n \sup_x |x^\alpha D^\beta \varphi| < \infty \right\}$$

$$= \left\{ \varphi \in C^\infty(\mathbb{R}^n) : \forall k \in \mathbb{N} \sum_{|\alpha| \leq k} \sup_x |\langle x \rangle^k D^\alpha \varphi| < \infty \right\}$$

- Some examples:

A. $C_c^\infty(\mathbb{R}^n)$

B. $f(x) = e^{-|x|^2}$

Some Properties of $\mathcal{S}$

- $\mathcal{S} \subset L^p$ for all $p \in [1,\infty]$, dense for $p \in [1,\infty)$
- Countable family of norms $\|\varphi\|_k := \sum_{|\alpha|\le k} \sup_x |\langle x\rangle^k D^\alpha \varphi|$ which induce a metric $d(\varphi,\psi) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|\varphi-\psi\|_k}{1+\|\varphi-\psi\|_k}$, so that $(\mathcal{S}, d)$ is complete
- Equivalent convergence: $\varphi_j \to \varphi$ in $\mathcal{S}$ iff $\|\varphi_j - \varphi\|_k \to 0 \quad \forall k$

The Fourier Transform On $\mathscr{S}$

- Given $f \in L^1$, define $\mathscr{F}f(\xi) := (2\pi)^{-n/2} \int f(x) e^{-ix\cdot\xi}\, d\xi$
- Fourier transform *intertwines* differentiation and multiplication: for $\varphi \in \mathscr{S}$

$$\mathscr{F}(D_j\varphi)(\xi) = \xi_j(\mathscr{F}\varphi)(\xi), \quad \mathscr{F}(x_j\varphi)(\xi) = -D_j(\mathscr{F}\varphi)(\xi)$$

- $\mathscr{F} : \mathscr{S} \to \mathscr{S}$, and it is continuous, bijective, and unitary on L^2 (after bounded linear extension)
- Fourier inversion formula:

$$f(x) = (2\pi)^{-n/2} \int \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi$$

What Are Schwartz Functions Good For?

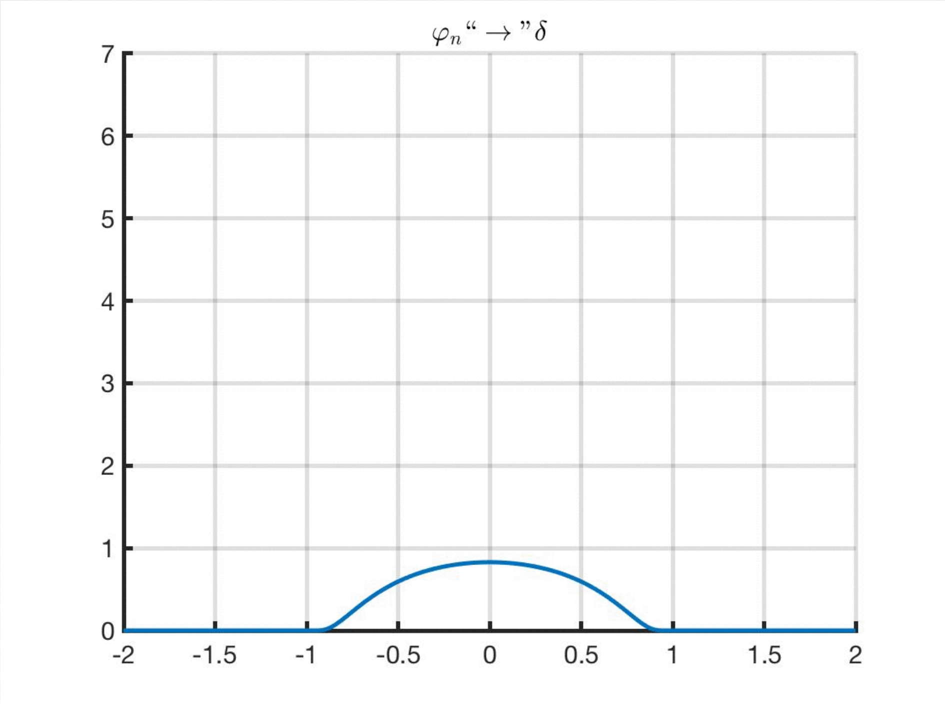
- Dense in lots of the important spaces (e.g. L^2), allow for formal calculations (IBP)
- Natural space to study Fourier transform, which is a powerful tool for solving linear PDEs in free space (PDE→ODE)

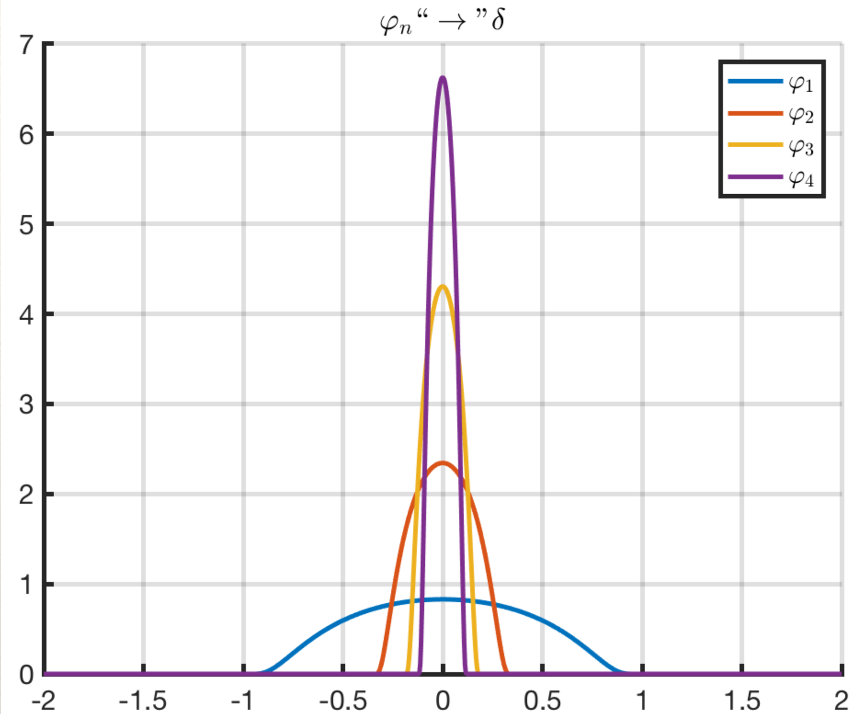
Application of Schwartz Functions: A Helmholtz Equation

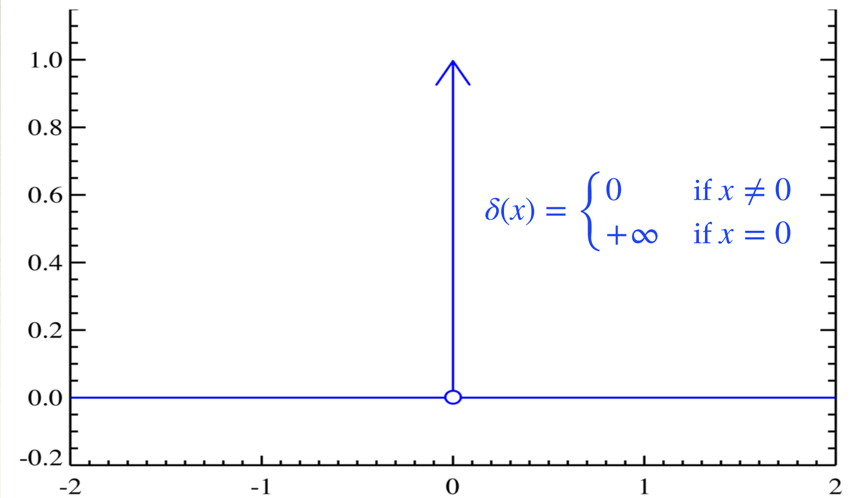
$$(I - \Delta)\varphi = \psi, \qquad \varphi, \psi \in \mathcal{S}$$

Idea Behind Distributions

- Loose def. of δ comes from "pointwise approximation"
- How to make sense of this convergence?
- Should emphasize "local properties" of functions







Tempered Distributions: $\mathcal{S}'$

- Tempered distributions:

$$\mathcal{S}' = \{u : \mathcal{S} \to \mathbb{C} : \ u \text{ is linear and continuous}\}$$

- Continuity: $\exists_C \exists_k \forall_{\varphi \in \mathcal{S}} \ |\langle u, \varphi \rangle| \leq C \|\varphi\|_k \iff \langle u, \varphi_j \rangle \to \langle u, \varphi \rangle$ if $\varphi_j \to \varphi$ in $\mathcal{S}$
- Topology: $u_j \to u$ in $\mathcal{S}'$ iff $\langle u_j, \varphi \rangle \to \langle u, \varphi \rangle$ for all $\varphi \in \mathcal{S}$
- Some examples
  - Schwartz functions: $\langle T_\psi, \varphi \rangle = \int \psi(x)\varphi(x)\, dx$
  - L^p functions: $\langle T_f, \varphi \rangle = \int f(x)\varphi(x)\, dx$
  - Dirac delta: $\langle \delta_a, \varphi \rangle = \varphi(a)$

Some Operations on $\mathscr{S}'$

- Motivating the distributional derivative: for $\varphi, \psi \in \mathscr{S}$

$$\langle \psi', \varphi \rangle = \int_{-\infty}^{\infty} \psi'(x)\varphi(x)\, dx = -\int_{-\infty}^{\infty} \psi(x)\varphi'(x)\, dx = -\langle \psi, \varphi' \rangle$$

- Differentiation: $\langle D^\alpha u, \varphi \rangle = (-1)^{|\alpha|}\langle u, D^\alpha \varphi \rangle$
- Multiplication by a smooth function with polynomial growth in all derivatives:

$$\langle fu, \varphi \rangle := \langle u, f\varphi \rangle$$

  - Ex. $\langle x_j u, \varphi \rangle = \langle u, x_j \varphi \rangle$

Distributional Derivative Example: Heaviside

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_0^\infty \varphi'(x)\, dx = \varphi(0) = \langle \delta, \varphi \rangle$$

$$H' = \delta$$

The Fourier Transform on $\mathscr{S}'$

- Defined by duality: $\langle \mathscr{F} u, \varphi \rangle := \langle u, \mathscr{F} \varphi \rangle$
- $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ continuously and bijectively, and its adjoint is still its inverse
- Still intertwines everything by duality
- "General" distributions don't play nice with $\mathscr{F}$
- Example: Fourier representation of δ

$$\langle \mathscr{F} \delta, \varphi \rangle = \langle \delta, \mathscr{F} \varphi \rangle = \mathscr{F} \varphi(0) = \langle (2\pi)^{-n/2}, \varphi \rangle$$

$$\implies \mathscr{F} \delta = (2\pi)^{-n/2}$$

$$\implies \delta = (2\pi)^{-n/2} \mathscr{F} 1$$

What Are Tempered Distributions Good For?

- Solving PDEs in a *weak sense*
  - L^2-based Sobolev spaces: $H^s = \left\{u \in \mathcal{S}' : \langle \xi \rangle^s \hat{u} \in L^2\right\}$
- Notion of weak solution is natural setting for finite element methods (caveats: use "general" distributions defined on compact domains)

Revisiting The Helmoltz Equation in $\mathscr{S}'$

Recall PDE $(I-\Delta)\varphi = \psi, \qquad \varphi, \psi \in \mathscr{S}$

Solution: $\varphi = \mathscr{F}^{-1}\left(\langle\xi\rangle^{-2}\mathscr{F}\psi\right)$

Some Resources

- Michael Taylor: "*Partial Differential Equations I: Basic Theory*" (chapter 4)
- F.G. Friedlander and M. Joshi: "*Introduction to the Theory of Distributions*"
- Lars Hörmander: "*The Analysis of Linear Partial Differential Operators I*"
- Laurent Schwartz: "*Théorie des distributions*"
- Robert Strichartz: "*A Guide to Distribution Theory and Fourier Transforms*"
- Walter Rudin: "*Functional Analysis*" (chapters 6-7)
- Gerald Folland: "*Real Analysis: Modern Technique and Their Applications*" (chapters 8-9)