INFINITE-DIMENSIONAL CENTER MANIFOLD THEORY WITH APPLICATIONS TO NONLINEAR PDES

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ABSTRACT. The center manifold theorem in the context of ordinary differential equations provides a manner of discussing the local behavior of a system near non-hyperbolic equilibria. This becomes a natural place to extend bifurcation theory past one dimension. A logical desire is to broaden such a theorem to the realm of partial differential equations, where we now need an infinite-dimensional analogue. Such a theorem would allow for dynamical approaches to PDEs, even in scenarios where the spectrum does not admit the generation of a strongly continuous semi-group. We will discuss a version of such a result and provide a concrete application to a semilinear elliptic equation on an unbounded cylindrical domain, which arises in the study of permanent waves in water channels of varying density.

CONTENTS

1. INTRODUCTION

When attempting to analyze the local behavior of a system of ordinary differential equations in the vicinity of an equilibrium point, one typically constructs invariant manifolds, each possessing different growth/decay properties, to study the flow on. These are split into the stable, unstable, and center manifolds, analogous to the stable, unstable, and center subspaces of a linear system (they turn out to be tangent to their counterparts). The stable and unstable manifolds are characterized by exponential decay and growth, respectively, but the center manifold is not characterized similarly. In fact, it allows for practically anything except for exponential growth or decay.

The occurrence of the center manifold is predicated on the existence of non-hyperbolic equilibria (equilibria where the corresponding linearized system has a matrix with eigenvalues with zero real part). The lack of non-hyperbolic equilibria allows for fairly straightforward local analysis, whereas presence of such equilibria requires the use of the local center manifold theorem to provide a reduction to analyze the flow on a system of lower dimension. Along with providing understanding of behavior near non-hyperbolic equilibria, the center manifold is a natural place to study bifurcations, where the real part of the eigenvalues of the linearized system cross the imaginary axis. Additionally, it tells us something given instability: if a fixed point changes from stable to unstable by varying a parameter, then nearby solutions are exponentially attracted to the center manifold. That is, all of the interesting, non-trivial, dynamics occur on the center manifold.

Many semilinear partial differential equations can be related to ODE theory by recasting the PDE as an evolution equation. That is, we can express it as a Cauchy problem of the form

$$\begin{cases} \dot{u}(t) = Au(t) + f(u(t), t), \quad t > 0 \\ u(0) = u_0 \end{cases}$$

defined on a Banach space X, with $u_0 \in X$, $A : D(A) \subset X \to X$ (typically) unbounded and linear, $f : X \times [0, T] \to X$. We typically also assume that A is densely-defined and $\rho(A) \neq \emptyset$. Since a solution $u(t)$ lives in $D(A) \subset X$ to each $t \in [0, T]$, and X is generically an infinite-dimensional space (such as L^2) rather than simply $\mathbb{R}^n$, our problem is now infinite-dimensional. A more concrete way to view such a change in dimension is through the following example (which has a secondary utility, as will be seen):

Example: Consider the heat equation $\partial_t u = \partial_x^2 u$ on $[0, \pi] \times (0, \infty)$, with initial data f and zero-Dirichlet boundaries. By first attacking the eigenvalue problem $\partial_x^2 u = \lambda u$, we arrive at a solution

$$u(x, t) = \sum_{j=1}^{\infty} u_j(t) \sin\left(\sqrt{\lambda_j} x\right),$$

where u_j solves

$$\dot{u}_j = \lambda_j u_j.$$

So, we have a countably-infinite number of ODEs to solve (each corresponding to a Fourier coefficient), and we can view our problem as looking for solutions $\hat{u} = (u_1, u_2, \cdots)$ to the problem $\frac{d}{dt}\hat{u} = \Lambda \hat{u}$.

So, we can convert some PDEs into "infinite-dimensional" ODEs. The generation of a flow is less automatic now than in the finite-dimensional setting, due to the increased size of the spectrum. To obtain a unique (mild) solution to a Cauchy problem for $u_0 \in D(A)$ with continuous non-linear term locally Lipschitz in X, one requires that the operator be the infinitesimal generator of a C_0-semigroup, which is equivalent to requiring that the spectrum lie in some left half-plane ([2]). In particular, this excludes operators with spectrum that is unbounded to the left and right of the imaginary axis. The difficulty of appropriately choosing the domain of the operator to allow for appropriate spectrum, regularity, etc. makes semigroup theory challenging to apply. The power of the center manifold theorem is that it allows us to circumvent many of these issues and still make significant qualitative conclusions.

2. The Infinite Dimensional Center Manifold Theorem

In the local center manifold reduction in ODEs, we took our system and broke it into the diagonal form

$$\begin{cases} \dot{u}_c = B_c u_c + \tilde{g}_c(u_c, u_s, u_u) \\ \dot{u}_s = B_s u_s + \tilde{g}_s(u_c, u_s, u_u) \\ \dot{u}_u = B_u u_u + \tilde{g}_u(u_c, u_s, u_u) \end{cases} \tag{2.1}$$

with B_c, B_s, B_u having eigenvalues of 0, negative, and positive real part, respectively, and $\tilde{g}_c, \tilde{g}_s, \tilde{g}_u \in C^{k+1}$ with no constant or linear terms. The local center manifold theorem guaranteed the existence of a map $h_c \in C^k$ defined on a neighborhood U of $u_c = 0$ so that the C^k-manifold

$$W^c_{loc} = \{u = u_c + h_c(u_c) : u_c \in U, u_s + u_u = h_c(u_c)\}$$

is invariant under the flow of the original system. Further, the flow on the center manifold was determined by

$$\dot{u}_c = B_c u_c + \tilde{g}_c(u_c, h_s(u_c), h_u(u_c)),$$

where $h_c = (h_s, h_u)$.

We will proceed similarly to this above, but with a few caveats. We split the equations as previous, but we add in the following assumptions:

(1) u_c is of finite dimension.
(2) There exist $\beta_+ > 0, \beta_- < 0$ so that $\{e^{tB_s}\}_{t\geq 0}$ and $\{e^{tB_u}\}_{t\leq 0}$ define C_0-semigroups in some space $\ell_{2,\theta}$ so that

$$\left\|e^{tB_u}\right\|_{\ell_{2,\theta}\to\ell_{2,\theta}} \leq Me^{\beta_+ t} \quad \forall t \leq 0$$
$$\left\|e^{tB_s}\right\|_{\ell_{2,\theta}\to\ell_{2,\theta}} \leq Me^{\beta_- t} \quad \forall t \geq 0$$

(3) $\tilde{g}_c, \tilde{g}_u, \tilde{g}_s \in C^{k+1}(\ell_{2,\theta}, \ell_{2,\theta})$

Here, $\ell_{p,\theta}$ is a weighted sequence space consisting of the Fourier coefficients of u (this is a natural space to consider, as our heat equation example illustrates), equipped with the ℓ_p norm weighted by the Japanese bracket of the summation index. By definition, $\ell_{2,\theta} \cong H^\theta_p = \overline{C^\infty_p}$, where the p denotes functions 2π periodic in each argument that allow for smooth extensions to $\mathbb{R}^n$ and the closure is in the H^θ-norm.

Remark. The assumptions on the nonlinear term and the semigroups can be weakened, and we can allow for an infinite dimensional center manifold if we assume the existence of $\{e^{tB_c}\}$ (that is, if we make the semigroup assumption). This entire procedure can be made significantly more general and abstract, as in [4]. The results of the author are weaker than hypotheses of C_0-semigroups, analytic semigroups, and more broad spectral assumptions.

As the name suggests, the local center manifold theorem is local in nature, so we need to apply a cutoff to obtain a neighborhood about $u_c = 0$. More precisely, we fix a radial cutoff $\chi \in C^\infty_0$ so that $\chi \equiv 1$ in $B_1(0)$ and $\chi \equiv 0$ outside $B_2(0)$, then replace $\tilde{g}$ in (2.1) by $g(u) := \tilde{g}_\rho(u) := \tilde{g}(u)\chi(\rho^{-1}\|u\|_{\ell_{2,\theta}})$, which satisfies $g \in C^{k+1}(\ell_{2,\theta}, \ell_{2,\theta})$ and $g(0) = 0, Dg(0) = 0$.

Now, we are prepared to state a version of the local center manifold theorem that is applicable to infinite dimensional problems, as presented in [3].

Theorem (Local Center Manifold Theorem)**.** Let $\eta \in (0, \beta)$, with $\beta = \min\{-\beta_-, \beta_+\}$. Then, there exists $\delta_0 > 0$ such that for g with $\sup_{u\in\ell_{2,\theta}} \|Dg(u)\|_{\ell_{2,\theta}} \leq \delta_0$, the center manifold is a C^k-manifold in $\ell_{2,\theta}$ invariant under the flow of (2.1) and tangent to the center subspace that can be characterized as

$$W^c_{loc} = \{u_0 \in \ell_{2,\theta} : \sup_{t\in\mathbb{R}} e^{-\eta|t|}\|u(t, u_0)\|_{\ell_{2,\theta}} < \infty\}$$

for η small enough, and it is unique. More precisely, there exists a neighborhood $U \subset \ell_{2,\theta}$ of $u_c = 0$ and a k-times differentiable function h sending u_c to (u_s, u_u) so that $W^c_{loc} = \{u_c + h(u_c) : u_c \in U\}$.

Remark. The uniqueness claim is true *given* an appropriate cut-off. However, the neighborhood is not independent of χ, so the manifold is not truly unique (just as in the finite dimensional case).

This theorem allows for problems with unbounded spectrum to both sides of the imaginary axis, making it applicable in cases where semigroups fail (solving problems of this nature was first studied by Kirchgässner in [1]). A sketch of a majority of the proof is provided in [3], and the full details of the more general, abstract version can be found in [4].

3. An Application to Elliptic Equations on Unbounded Cylindrical Domains

Here, we will provide a simple example in the form of an elliptic equation on a strip. Applying center manifold theory to elliptic equations through spatial dynamics was first used by Kirchgässner [1], who was studying permanent waves in density-stratified channels. Additional examples can be found in [4].

Consider the elliptic equation

$$\begin{cases} \Delta u &= \alpha u - u^3, \qquad (x,y) \in \mathbb{R} \times (0,\pi) \\ u\big|_{y=0,\pi} &= 0 \end{cases}.$$

The idea of spatial dynamics, as done by Kirchgässner in [1], is to view the x variable (the unbounded variable) as being the time variable and translate this to an evolution equation. This sort of approach is also taken in [4]. We will proceed as in [3]. We will be searching for bounded solutions.

Expanding u in Fourier series (we are working in $\ell_{2,\theta}$ spaces and the y domain is bounded), we write

$$u(x,y) = \sum_{m=1}^{\infty} u_m(x) \sin(my).$$

Plugging this into the equation and integrating against $\sin(ny)$ gives

$$(\partial_x^2 - n^2) u_n(x) = \alpha u_n + g_n(u_1, u_2, \cdots),$$

where

$$g_n = -\frac{2}{\pi} \int_0^{\pi} (u(x,y))^3 \sin(ny)\, dy = \mathcal{O}(\|u\|_{\ell_3}^3) \text{ as } u \to 0,$$

which forms a system

$$\begin{pmatrix} \dot{u}_n(x) \\ \dot{v}_n(x) \end{pmatrix} = \begin{pmatrix} n & 1 \\ \alpha & -n \end{pmatrix} \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} + \text{H.O.T.}$$

Linearizing about $u = 0$, we find eigenvalues

$$\lambda_{n\pm} = \pm\sqrt{n^2 + \alpha},$$

which is the entirely of the spectrum.

Note that if $\alpha = -1$, then 0 is an eigenvalue of multiplicity two, meaning that we will have a two-dimensional local center manifold (see Figure 1 for the spectrum). We are interested in $\alpha \approx -1$, so this will persist, but for larger negative perturbations, the dimension will increase by an even amount. For $\alpha \approx -1$, the eigenvalues (starting past $n = 1$) form two real sequences going to $\pm\infty$. So, the spectrum is unbounded to both the left and the right of the imaginary axis, making semigroup theory non-viable. We can, however, apply center manifold theory.

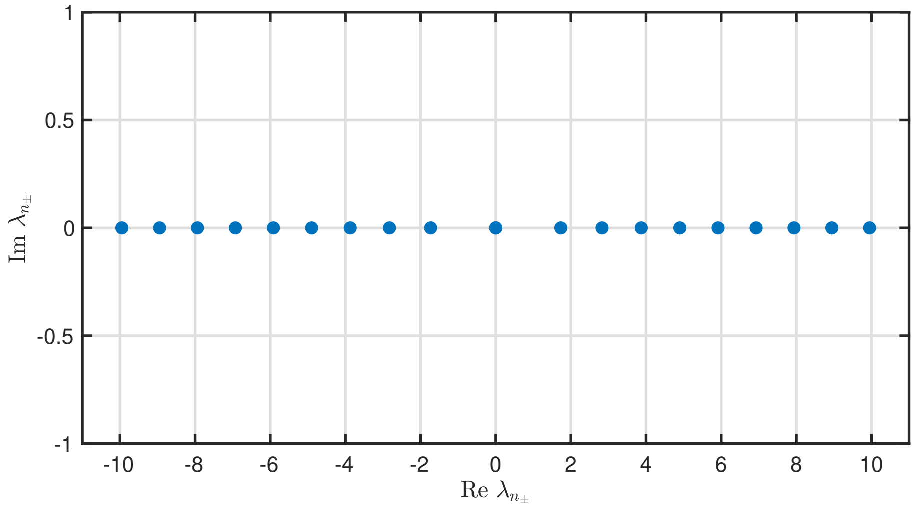

FIGURE 1. Spectrum with $\alpha = -1$

The first assumption is automatically satisfied since there is finite spectrum (two eigenvalues) on the imaginary axis. Next, we must analyze the remaining stable and unstable problems. To diagonalize, re-define v_n by $\dot{u}_n = \lambda_n v_n$. Then, calling $w_n = u_n + v_n$, $w_{-n} = u_n - v_n$ gives the system

$$\begin{pmatrix} \dot{w}_n(x) \\ \dot{w}_{-n}(x) \end{pmatrix} = \begin{pmatrix} \lambda_n & 0 \\ 0 & -\lambda_n \end{pmatrix} \begin{pmatrix} w_n(x) \\ w_{-n}(x) \end{pmatrix} + \begin{pmatrix} g_n \\ -g_n \end{pmatrix}.$$

The nonlinearity is smooth from $\ell_{2,\theta}$ to itself for $\theta > 1/2$, so we can apply the center manifold theorem. Such smoothness can either be seen via Sobolev-type embeddings or, as in [3], noting that functions satisfying zero Dirichlet boundary conditions can be smoothly extended to be odd and 2π periodic in y. Then, since odd functions cubed are still odd, our problem can be recast in the subspace of odd, 2π-periodic functions in y, which is an invariant subspace. Smoothness will follow from this. The semigroup properties are obvious, taking $\beta_+ = \lambda_{2_+}, \beta_- = \lambda_{2_-}$.

Applying the theorem, we obtains functions $h_j(u_1, \alpha) = u_j$ for each $j \geq 2$ and get the reduction on the center manifold as

$$\ddot{u}_1 - u_1 = \alpha u_1 + g_1(u_1, h_2(u_1, \alpha), h_3(u_1, \alpha), \cdots),$$

or (directly computing the lowest order g_1 term)

$$\ddot{u}_1 = (\alpha + 1)u_1 - \frac{3}{4}u_1^3 + \mathcal{O}(u_1^5).$$

After dropping the higher-order non-linear terms, this can be written equivalently as the system

$$\begin{cases} \dot{v} = w \\ \dot{w} = (\alpha + 1)v - \frac{3}{4}v^3 \end{cases}$$

When $0 < \alpha + 1 \ll 1$, this system possesses two homoclinic orbits, which correspond to spatially localized solutions to the original problem (see Figure 2 on the subsequent page).

Hence, center manifold theory provided valuable qualitative information in a circumstance where semigroup theory was non-applicable.

Remark. The dropping of the $\mathcal{O}(u_1^5)$ terms can be justified by reversibility. Indeed, one can choose the cutoff so that it preserves the $(x, u) \mapsto (-x, u)$ symmetry in the equation. Then, the reduced system will be invariant with respect to the reflection $(x, u_1, v_1) \mapsto (-x, u_1, -v_1)$, meaning that the phase portrait will still be symmetric under reflections about the u_1-axis. This implies that the homoclinic orbits will not break up.

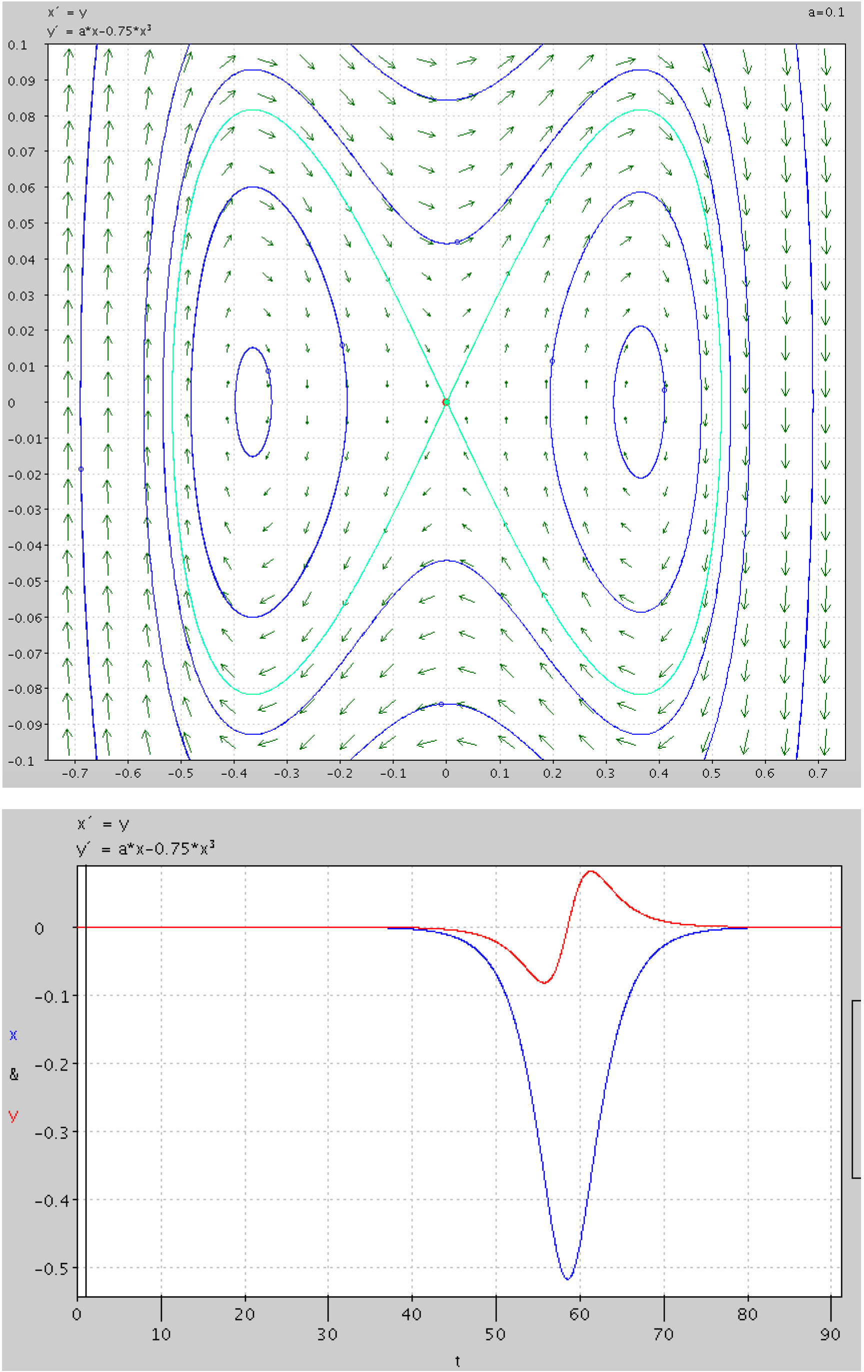

FIGURE 2. (Top) Phase portrait in $u_1 - u_1'$ space, $\alpha + 1 = 0.1$
(Bottom) Evolution of u_1 (blue) and u_1' (red) in x for $\alpha + 1 = 0.1$

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