Wave Equations, Local Energy Decay, and Trapping

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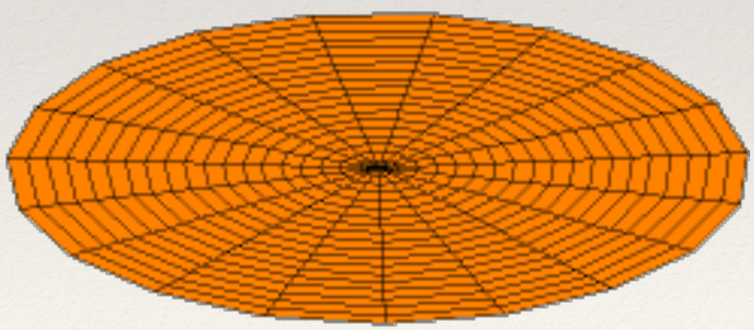
The Wave Equation

- Let $\square = \dfrac{\partial^2}{\partial t^2} - \Delta, \qquad \Delta = \displaystyle\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ $\leftarrow$ "Laplacian"
- Wave equation on $\mathbb{R}^+ \times \mathbb{R}^n$:

$$\begin{cases} \square u = 0 \\ u(0,x) = f(x) \\ \partial_t u(0,x) = g(x) \end{cases}$$

- Helpful for modeling many physical phenomena
  - Fluids
  - Acoustics
  - Electromagnetic waves
    - Quantum mechanics
    - General relativity



Variant: The Damped Wave Equation

- Recall damping force $-\alpha u'$ from ODE mass-spring problem:

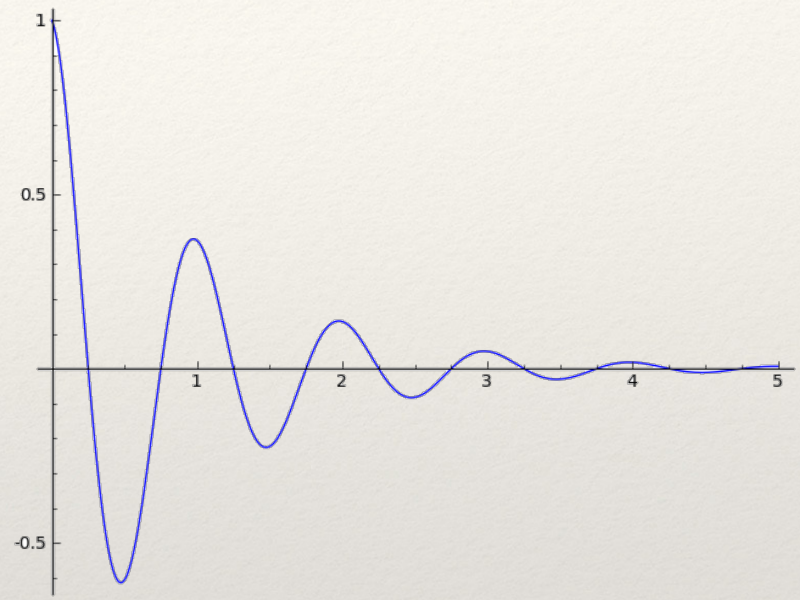
$$\begin{cases} mu'' + \alpha u' + ku = 0 \\ m, \alpha > 0,\ k \geq 0 \end{cases}$$

$$\alpha = \text{damping coefficient}$$

- Force moves in opposite direction of velocity
- Example of a damping force: air resistance
- The damped wave equation on $\mathbb{R}^+ \times \mathbb{R}^n$:

$$\begin{cases} \square u + a(x)\partial_t u = 0 \\ u(0,x) = f(x) \\ \partial_t u(0,x) = g(x) \end{cases}$$

where a is smooth, non-negative, and positive on an open set.



Energy Conservation

- Suppose that
  - $\Box u = 0, \qquad \Box v + a\partial_t v = 0$
  - For every t, $u(t,x)$ and $v(t,x)$ are zero for large $|x|$
- Define $E[w](t) = \dfrac{1}{2}\displaystyle\int |\partial_t w|^2 + |\nabla w|^2\, dx$
- $\forall_{t>0} E[u](t) = E[u](0)$, and $E[v](t)$ is decreasing in t
- Proof: Take the derivative of $E[u]$ in t. Equivalently, integrate $\partial_t u \Box u$ in space and time, then integrate by parts. Same for v.
- Conclusions:
  - Solutions to the wave equation keep the same energy.
  - Solutions to the damped wave equation lose energy.
- We'll focus on the wave equation for now.

Local Energy Decay for the Wave Equation

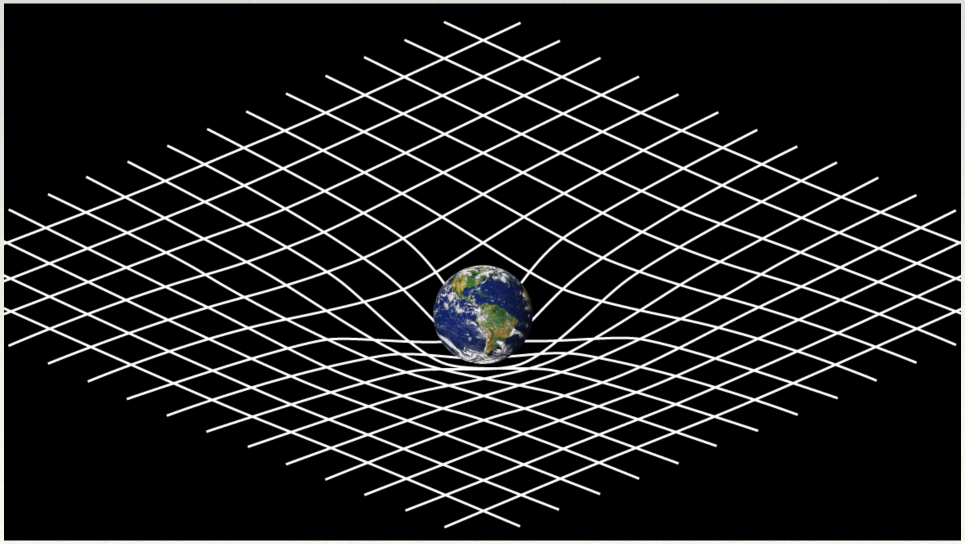
- Although energy is conserved for solutions to the wave equation, it will decay within bounded spatial sets.
- Under the previous assumptions, solutions satisfy

$$\sup_{R>0} \int_0^\infty \int_{|x|\leq R} |R^{-1/2}\partial_t u|^2 + |R^{-1/2}\nabla u|^2 + |R^{-3/2}u|^2 \, dxdt \leq CE[u](0)$$

  provided $n \geq 3$. This is called a local energy decay estimate.
- This is proved by multiplying $\Box u$ by something more complicated and doing clever integration by parts.
- Utility:
  - Important measure of dispersion (how the wave spreads out over time)
  - Can aid in obtaining long-time existence for nonlinear problems

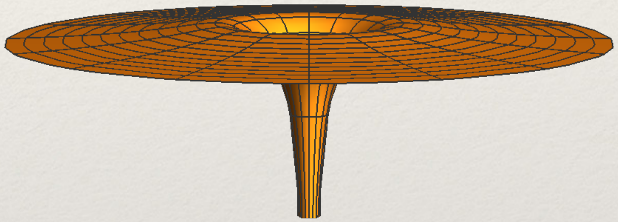
Adding Spacetime Geometry

- Many applications include background geometry.
- We will consider the spacetime $\mathbb{R}^4$, but we'll let it be curved (have a metric tensor). This forms $\mathbb{R}^4$ into a Lorentzian manifold.
- One must adapt the equations to the geometry
  - Ex. $\Delta_g = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} g^{ij}(x) \frac{\partial}{\partial x_j}$
  - g^{ij} = components of the inverse metric tensor g^{-1}
  - For the original wave equation, $g^{-1} = g = I$
- Assumption: metric is asymptotically flat
- Important result: If the metric is very close to being flat, then LED holds.
- Question: What about larger variations?



Obstructions to LED

1. Trapping: null geodesics (paths in spacetime that light follow) stay in a compact set
   - Example: blackholes
   - Trapping implies no LED



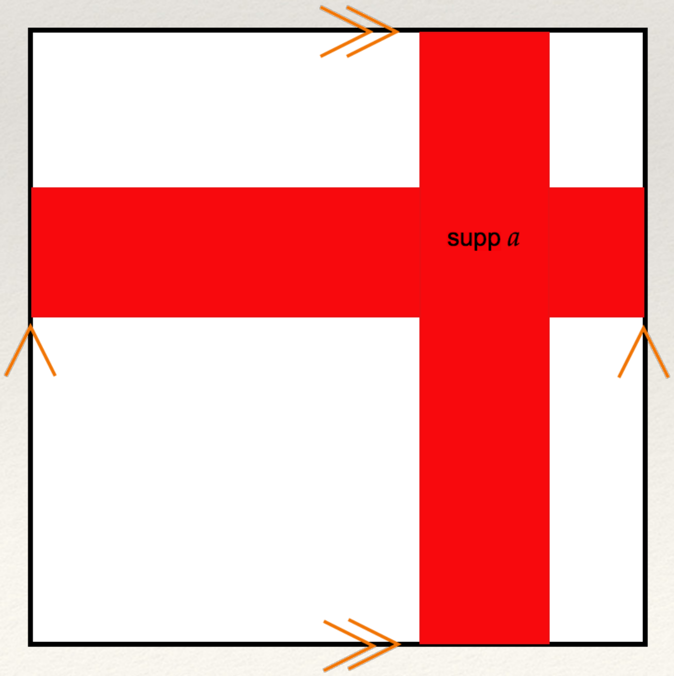
2. Bad spectrum
   - Eigenvalues whose eigenvectors (eigenfunctions) have growth

These turn out to be the **only** obstructions for (stationary) asymptotically flat problems (ref: Metcalfe, Sterbenz, Tataru '20)! Since trapping backgrounds are physically relevant, we'd like to remove the non-trapping hypothesis.

Back to Damping

- Hope: damping "removes energy," so (maybe) we can allow for trapping and still get LED.
- Issue: what if the damping never affects trapped trajectories?
- New assumption: the geometric control condition

  "every bounded null geodesic intersects $\{x : a(x) > 0\}$ in finite time"
- Mathematically, trapping comes into play for waves with high frequencies. One must prove an estimate using another multiplier argument. These multipliers are called pseudodifferential operators and are constructed using tools from microlocal analysis.



Geometric control on the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$