Scattering, Resonances, and Wave Equations

Collin Kofroth

GMA Visions Seminar
February 23, 2021

About Me

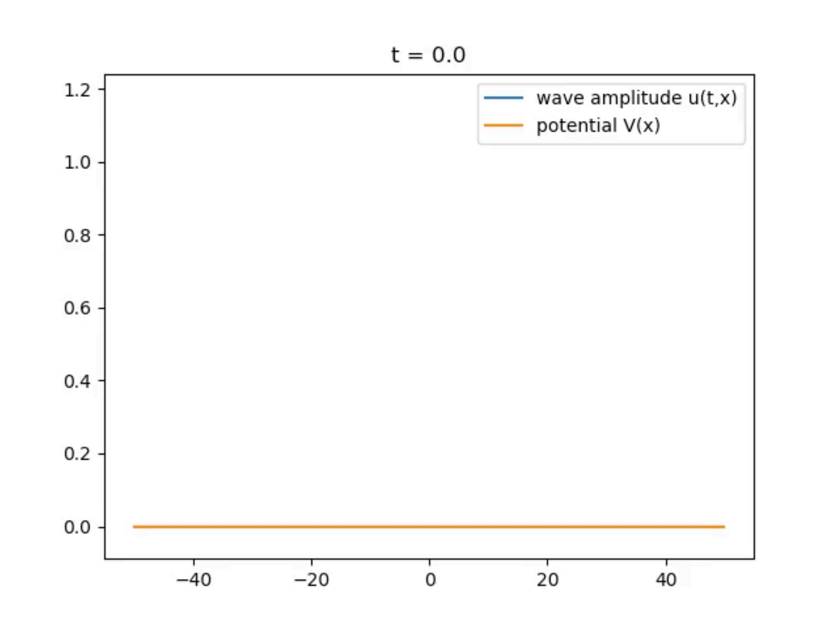
- 4th year
- Advisor: Jason Metcalfe
- Area of research: analysis of PDEs
  - Specific area: Local energy decay of damped waves on asymptotically flat space-times
- Comps taken: Analysis, geo/topo, sci comp
- First year courses relevant to research: analysis, geo/topo, some tools from methods of applied math
- Other important tools: PDEs (obv), functional analysis, spectral theory, microlocal analysis

Why Scattering?

- Utilizes some first-year analysis (especially complex analysis - I'll point out where explicitly)
- Courses important to study scattering: real/complex analysis, functional analysis, PDEs
- Lively area of research within analysis of PDEs
  - Significant ties to semiclassical analysis (big here!)
- Relevant area within department: talk to Jeremy Marzuola, Hans Christianson

What is Scattering Theory?

- What happens when a particle or wave encounters something that may force it to deviate from its original behavior?
- Three main types
  - Obstacle scattering (common perspective for classical scattering): what happens when a particle hits something?
  - **Potential scattering** (common perspective for quantum scattering): what happens when a particle (or wave) encounters a potential?
  - Geometric scattering: what happens if we perturb the geometry (metric)?
- Applications in
  - Quantum mechanics
  - General relativity
  - Medical imaging (somehow)



Courtesy of **Peter Hintz**

Objects of Interest: Resonances

- Sort of like eigenvalues, but for equations on unbounded domains
  - Along with telling us about oscillation rates, resonances carry on info on decay rates.
  - Resonance expansion:

$$u(t,x) \sim \sum e^{-i\tau_j t} a_j(x), \qquad \tau_j \in \mathbb{C}$$

$$\tau_j = \text{resonances}$$

vs.

Fourier expansion on compact spatial domain:

$$u(t,x) \sim \sum e^{i\lambda_j x}(e^{i\lambda_j t} a_j + e^{-i\lambda_j t} b_j), \qquad \lambda_j \in \mathbb{R}$$

$$\lambda_j = \text{eigenvalues}$$

- They come from poles of an operator-valued function called the resolvent (really its meromorphic continuation).

Local Energy Decay

- Local energy functional for compact $K \subset \mathbb{R}^3$

$$E_K[u](t) = \int_K |\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2 \, dx$$

- This will decay exponentially, according to our expansion (provided each resonance lies in the lower half plane). This occurs because $\mathbb{R}^3$ is not compact. Energy is escaping to infinity!
- In the compact setting, solutions do not decay exponentially, and energy is constant.

Some Notation

- $L^2(\mathbb{R}^3) = \{f : \mathbb{R}^3 \to \mathbb{C} \mid \int_{\mathbb{R}^3} |f(x)|^2 \, dx < \infty\}$
- $H^2(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) \mid \partial^\alpha f \in L^2(\mathbb{R}^3) \text{ for all } |\alpha| \leq 2\}$
- $L^\infty(\mathbb{R}^3) = \{f : \mathbb{R}^3 \to \mathbb{C} \mid \exists M \text{ s.t.} |f| \leq M \text{ a.e.}\}$
- I'll say $f = \mathcal{O}(g) \iff (\exists C)(\forall x \gg 1) |f(x)| \leq C|g(x)|$
- c subscripts denote compact support, *loc* subscripts mean that elements are in the un-subscripted space for every compact set
- Any time I say "operator," I mean bounded (i.e. continuous) linear operator.

Model Problem: Wave Equation

- Wave equation with potential:

$$\begin{cases} \left(\partial_t^2 - \Delta + V\right) u = 0, \qquad V \in L_c^\infty(\mathbb{R}^3, \mathbb{C}), \qquad \Delta = \sum_{j=1}^{3} \partial_{x_j}^2 \\ u(0) = 0, \qquad \partial_t u(0) = f \in L_c^2(\mathbb{R}^3) \\ u = 0 \text{ for } t < 0 \end{cases}$$

- Proceed formally: take the Fourier-Laplace transform in time

$$(-\tau^2 - \Delta + V)\hat{u}(\tau) = f, \qquad \operatorname{Im}\tau > 0$$

- Solve for $\hat{u}$ by inverted operator: $R(\tau) = (-\tau^2 - \Delta + V)^{-1}, \qquad \operatorname{Im}\tau > 0$

- Invert Fourier transform: $u(t) = (2\pi)^{-1} \displaystyle\int_{\operatorname{Im}\tau = M} e^{-i\tau t} R(\tau) f \, d\tau, \qquad M > 0$

- I *may* have cheated in two places:
  - When can I invert that operator?
  - Can I really invert the Fourier transform? And how do I use it to get the resonance expansion?

The Free Resolvent

- Starting point for operator inversion is our equation no potential: $(-\tau^2 - \Delta)u = f$
- Solve using Fourier analysis to get family of operators called the free resolvent

$$R_0(\tau) = (-\tau^2 - \Delta)^{-1} : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3), \qquad \operatorname{Im} \tau > 0$$

$$R_0(\tau)f(x) = \int_{\mathbb{R}^3} \frac{e^{i\tau|x-y|}}{4\pi|x-y|} f(y)\, dy$$

- As family of operators, this has analytic continuation to $\tau \in \mathbb{C}$, now as a function

$$R_0(\tau) : L^2_c(\mathbb{R}^3) \to H^2_{loc}(\mathbb{R}^3) \quad \text{i.e.} \quad \rho R_0(\tau)\rho : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3) \qquad \forall \rho \in C^\infty_c(\mathbb{R}^3)$$

- Connect back to wave equation: use Cauchy's theorem to deform the contour

$$u(t) = (2\pi)^{-3} \int_{\operatorname{Im} \tau = -M} e^{-i\tau t} R_0(\tau) f(\tau)\, d\tau = \mathcal{O}_{L^2(K)}(e^{-Mt}), \qquad \text{any } \; M > 0,\ K \text{ compact}$$

- Compare to Sharp Huygen's Principle: $u \equiv 0$ within any compact set $K \subset \mathbb{R}^3$ for all $t \geq T_K$

The Free Resolvent Continued: Closer Look at $\mathbb{R}$

- Consider the branch of the square root that cuts out the non-negative real axis and maps into the upper half plane and call $\widetilde{R_0}(\tau) = R_0\left(\sqrt{\tau}\right)$. Then,

$$u_\pm(x) = \lim_{\epsilon\to 0^+} \widetilde{R_0}(\tau \pm i\epsilon)f(x) = \int_{\mathbb{R}^3} \frac{e^{\pm i\sqrt{\tau}|x-y|}}{4\pi|x-y|} f(y)\,dy, \qquad \tau > 0, \quad f \in C_c^\infty(\mathbb{R}^3)$$

- Facts about $u_\pm$:
  - $(-\tau - \Delta)u_\pm = f$
  - $$u_\pm(x) \sim \frac{e^{\pm i\sqrt{\tau}|x|}}{|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right) \implies \begin{cases} u_\pm(x) & = \mathcal{O}\left(\frac{1}{r}\right) \\ (\partial_r \mp i\sqrt{\tau})u_\pm(x) & = \mathcal{O}\left(\frac{1}{r^2}\right) \end{cases}$$
- We call $u_\pm$ the outgoing/incoming solutions, and they are the unique solutions to our equation with these conditions.
- This is called the qualitative limiting absorption principle. Better estimates on such limits are related to the quantitative limiting absorption principle.

Fredholm Theory

- A Fredholm operator is a bounded linear map A between Banach spaces such that $\dim \ker A, \dim \operatorname{coker} A < \infty, R(A)$ closed.
- Fredholm operators are invertible modulo compact operators.
- Families of operators are analytic iff complex differentiable in norm iff they have a power series expansion. Meromorphicity is defined using Laurent series.
- Analytic Fredholm Theorem: If $\{A(z)\}_{z\in\Omega}$ an analytic family of Fredholm operators and $A(z_0)^{-1}$ exists for some z_0, then $z \mapsto A(z)^{-1}$ is a meromorphic family of Fredholm operators with poles of finite rank.

The Scattering Resolvent

- Add in the potential: $\underbrace{(-\tau^2 - \Delta + V)}_{P(\tau)} u = f, \qquad V \in L^\infty_c(\mathbb{R}^3, \mathbb{C})$
- The desired inverse of $P(\tau)$, denoted $R(\tau)$, is called the scattering resolvent.
- For large $\operatorname{Im} \tau$, $R_0(\tau)$ gives an approximate inverse to $R(\tau)$, so

$$R(\tau) = (-\tau^2 - \Delta + V)^{-1} : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3), \qquad \operatorname{Im} \tau \gg 1$$

- Using Fredholm theory, this has meromorphic continuation to $\tau \in \mathbb{C}$, now as a function $R(\tau) : L^2_c(\mathbb{R}^3) \to H^2_{loc}(\mathbb{R}^3)$.
- The poles of this continuation are called (scattering) resonances. I will assume that they are simple on the next slide.
- **Theorem**: $\tau \neq 0$ is a resonance iff $P(\tau)u = 0$ for some non-zero, outgoing u (outgoing in the sense before or, equivalently, in the range of $R_0(\tau)$).
- **Theorem**: if τ is not a resonance, then $u = R(\tau)f$ is the unique outgoing solution to

$$P(\tau)u = f, \qquad f \in L^2_c(\mathbb{R}^3)$$

Resonance Expansions

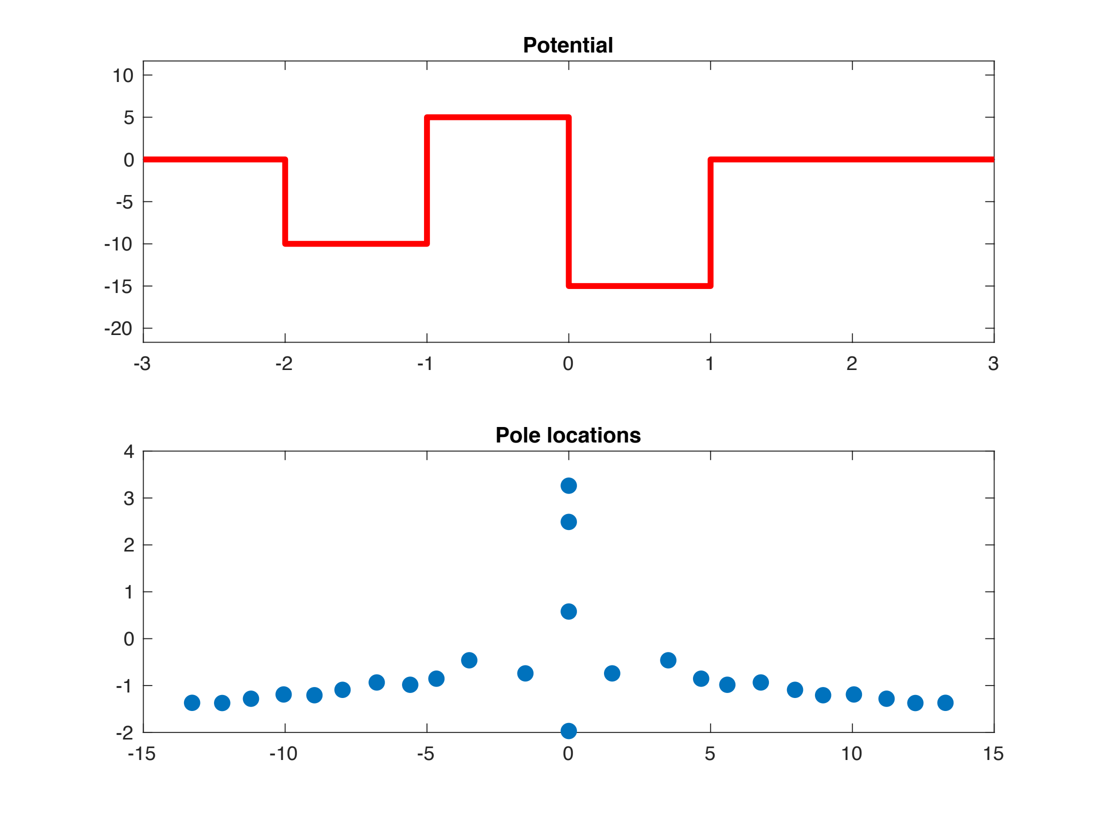
- For $\mathrm{Im}\ \tau$ large enough, we can make sense of our Fourier inversion as an element of $L^2_{loc}(\mathbb{R}^+, L^2(\mathbb{R}^3))$.
- Contour shifting needs more care, but one can get estimates within resonance-free regions and choose a witty contour to show that for any $A > 0$, we have a finite sum

$$u(t,x) = \sum_{\mathrm{Im}\ \tau_j > -A} e^{-i\tau_j t} a_j(x) + E_A(t), \qquad E_A = \mathcal{O}_{H^2_{loc}}(e^{-tA})$$

- $\tau_j =$ resonances, $a_j =$ resonant states, $e^{-i\tau_j t} a_j =$ residues
- $\mathrm{Re}\ \tau_j =$ rate of oscillation, $-\ \mathrm{Im}\ \tau_j =$ rate of decay

Resonances with Different Potentials

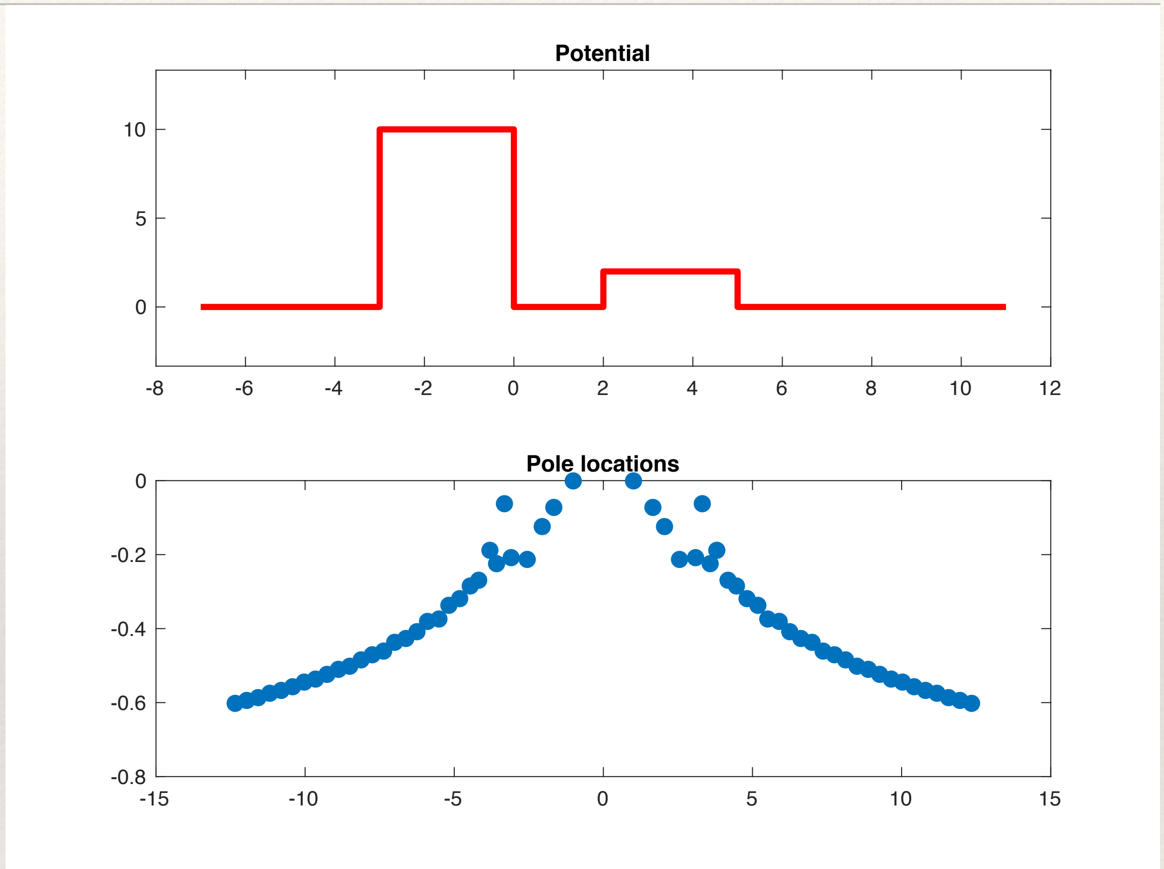
- Suppose that V is real-valued (generally what we care about).
    - Resonances with positive imaginary part are purely imaginary and their squares are eigenvalues of $-\Delta + V$.
    - There are no non-zero real resonances (Rellich).



Credit: **David Bindel**'s code

Resonances with Different Potentials

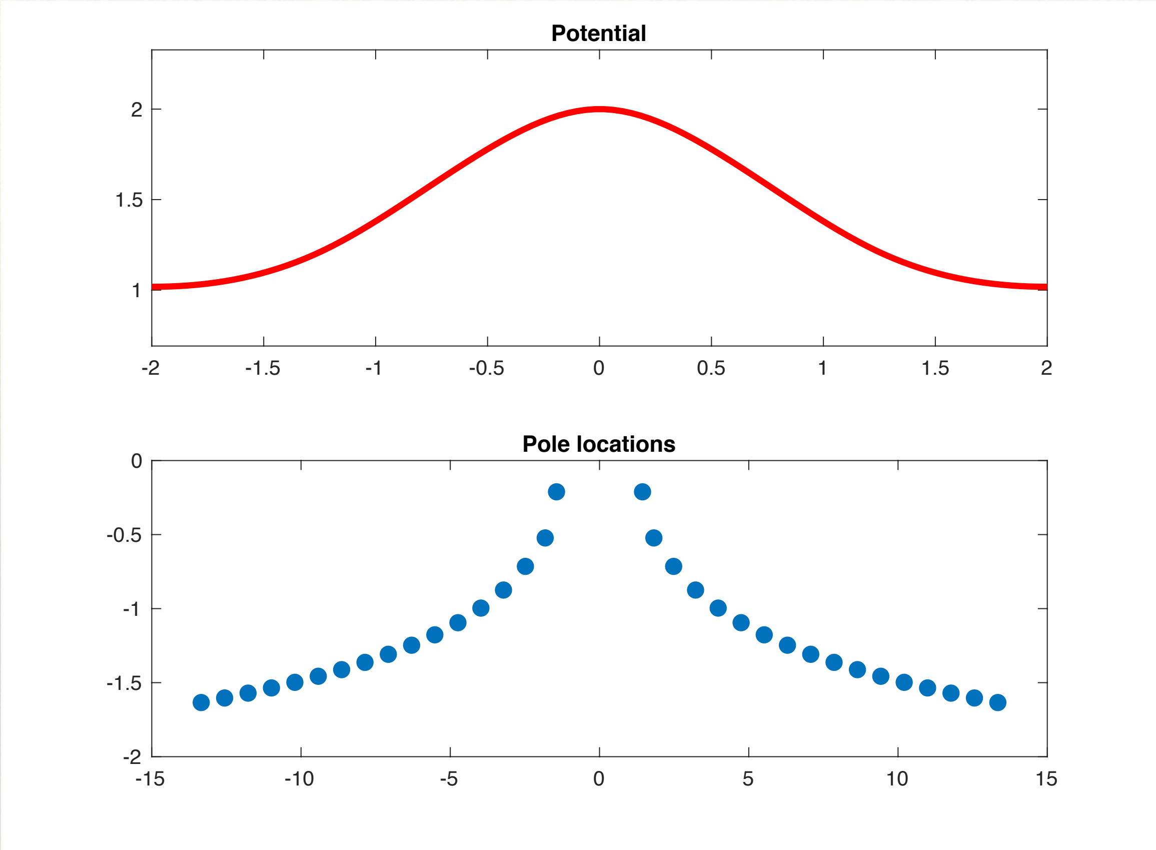
- If $V \geq 0$, then all non-zero resonances have negative imaginary part.



Credit: **David Bindel**'s code

Resonances with Different Potentials

- If $V \geq 0$ and $V > 0$ on an open set, then there are no real resonances.
- Via the resonance expansion, this implies exponential decay of the wave equation within compact sets.



Credit: **David Bindel**'s code

Resonance Expansions In Geometric Scattering

- De Sitter space: hyperbolic model of the universe ($\Lambda > 0$)
  - Follows "similar" general procedure
- Black holes, even those with sufficiently small angular momentum (Kerr, Kerr-de Sitter)
  - "Quasi-normal modes"
  - Follows "similar" general procedure
  - Hard
    - Trapping affects high frequency (large $|\mathrm{Re}\,\tau|$) estimates, loss of information
- References: see various papers by **András Vasy**, **Peter Hintz**, and **Semyon Dyatlov**

Other Areas of Interest with Resonances

- Counting them
  - Compare to Weyl law for counting eigenvalues
  - Very open area (asymptotics, lower bounds)
- Distribution of resonances / resonance-free regions
  - Related to resonance expansions
- Trace formulas
  - Related to counting resonances (e.g. lower bounds)

Some References

- **Dyatlov** and **Zworski** - *Mathematical Theory of Scattering Resonances*
- **Reed and Simon** - *Methods of Modern Mathematical Physics, Vol. 3: Scattering Theory*
- **Taylor** - *Partial Differential Equations II: Qualitative Studies of Linear Equations* (see Ch. 9 for scattering by obstacles)
- **Hörmander** - *The Analysis of Linear Partial Differential Operators II: Differential Operators with Constant Coefficients* (see Ch. XIV)