Local Energy Decay for Damped Waves

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The Wave and Damped Equations

- Let $\square = \dfrac{\partial^2}{\partial t^2} - \Delta, \qquad \Delta = \displaystyle\sum_{j=1}^{3} \dfrac{\partial^2}{\partial x_j^2}$

- Wave equation on $\mathbb{R}_+ \times \mathbb{R}^3$:

$$\begin{cases} \square\, u = 0 \\ u(0,x) = f(x) \\ \partial_t u(0,x) = g(x) \end{cases}$$

- Damped wave equation on $\mathbb{R}_+ \times \mathbb{R}^3$:

$$\begin{cases} \square\, u + a(x)\partial_t u = 0 \\ u(0,x) = f(x) \\ \partial_t u(0,x) = g(x) \end{cases}$$

where a is smooth and non-negative.

Global Energy Bounds

- Suppose that
  - $\Box u = 0, \qquad \Box v + a(x)\partial_t v = 0$
  - For every t, $u(t,x)$ and $v(t,x)$ are zero for large $|x|$
- Define $E[w](t) = \dfrac{1}{2}\displaystyle\int |\partial w|^2\, dx, \qquad \partial = (\partial_t, \nabla_x)$
- $(\forall t \geq 0) E[u](t) = E[u](0)$, and $E[v](t)$ is non-increasing in t
- Proof: Integrate $\partial_t u \Box u$ in space and time, then integrate by parts. Same for v.
- Conclusions
  - Solutions to the wave equation keep the same energy.
  - Solutions to the damped wave equation lose energy.

Focus on the wave equation for now

Local Energy Decay Heuristic I

Although energy is conserved for solutions to the wave equation, it will decay within compact spatial sets.

Local Energy Decay Heuristic II

Heuristic: If $\tilde{u}$ is a wave packet, then by finite speed of propagation,

$$\int_0^\infty \int_{|x|\le R} |\partial \tilde{u}|^2 \, dx dt \lesssim R \sup_t \int_{|x|\le R} |\partial \tilde{u}|^2 \, dx \lesssim R E[\tilde{u}](0)$$

$$\implies R^{-1/2} \|\partial \tilde{u}\|_{L^2_{t,x}([0,\infty)\times\{|x|\le R\})} \lesssim \|\partial \tilde{u}(0)\|_{L^2}$$

Morawetz Estimate I

- **Theorem** (Morawetz): If $n \geq 3$, then

$$\int_0^T \int_{\mathbb{R}^n} \frac{|\not\nabla u|^2}{r} \, dxdt \lesssim E[u](0),$$

where $\nabla = \frac{x}{r}\partial_r + \not\nabla$, $\partial_r = \frac{x}{r} \cdot \nabla$

- Proof idea for $n = 3$: Multiply $\Box u$ by $Qu := C\partial_t u + \partial_r u + \frac{1}{r}u$

$$\int_0^T \int_{\mathbb{R}^3} \Box u \partial_r u \, dxdt = \text{Time boundary terms} - \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \partial_r((\partial_t u)^2) \, dxdt + \int_0^T \int_{\mathbb{R}^3} \nabla u \cdot \nabla \partial_r u \, dxdt$$

$$= \text{Time boundary terms} + \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \partial_r(|\nabla u|^2 - (\partial_t u)^2) \, dxdt + \int_0^T \int_{\mathbb{R}^3} \nabla u \cdot [\nabla, \partial_r] u \, dxdt$$

$$= \text{Time boundary terms} + \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \partial_r(|\nabla u|^2 - (\partial_t u)^2) \, dxdt + \int_0^T \int_{\mathbb{R}^3} \frac{|\not\nabla u|^2}{r} \, dxdt$$

Morawetz Estimate II + Local Energy Decay I

- Next,

$$\int_0^T\int_{\mathbb{R}^3} \Box u \frac{u}{r}\, dxdt = \text{Time boundary terms} - \frac{1}{2}\int_0^T\int_{\mathbb{R}^3} \frac{\partial_r(u^2)}{r^2}\, dxdt + \int_0^T\int_{\mathbb{R}^3} \frac{|\nabla u|^2 - (\partial_t u)^2}{r}\, dxdt$$

$$\Longrightarrow\ 0 = \int_0^T\int_{\mathbb{R}^3} \Box u Qu\, dxdt = \text{Time boundary terms} + \int_0^T\int_{\mathbb{R}^3} \frac{|\nabla\!\!\!/ u|^2}{r}\, dxdt - \frac{1}{2}\int_0^T\int_{\mathbb{R}^3} \frac{\partial_r(u^2)}{r^2}\, dxdt$$

- If we take $Q = C\partial_t u + f(r)\partial_r u + \dfrac{n-1}{2r} f(r)u$, where $f(r) = \dfrac{r}{r+2^j}$, then we get

$$\sup_{j\geq 0}\left(\|\langle x\rangle^{-1/2}\partial u\|^2_{L^2_{t,x}([0,\infty)\times\{\langle x\rangle\approx 2^j\})} + \|\langle x\rangle^{-3/2} u\|^2_{L^2_{t,x}([0,\infty)\times\{\langle x\rangle\approx 2^j\})}\right) \lesssim E[u](0),$$

provided $n \geq 3$. This is called an integrated local energy estimate.

- Note: $\langle x\rangle := (1+|x|^2)^{1/2}$

Local Energy Decay for the Wave Equation II

- Utility:
  - Scattering theory
  - Important measure of dispersion
    - Global-in-time Strichartz estimates
      - $\|u\|_{L_t^p L_x^q} \lesssim \|f\|_{\dot{H}^r} + \|g\|_{\dot{H}^{r-1}}$ for admissible p, q, r
    - Pointwise decay estimates
      - $|u(t,x)| \lesssim_x t^{-\alpha}$
  - Long-time existence for nonlinear waves
    - E.g. $\Box u = Q(\partial u, \partial^2 u)$

Local Energy Decay for the Wave Equation III

- Recall estimate

$$\sup_{j\geq 0}\left(\|\langle x\rangle^{-1/2}\partial u\|^2_{L^2_tL^2_x([0,\infty)\times\{\langle x\rangle\approx 2^j\})}+\|\langle x\rangle^{-3/2}u\|^2_{L^2_tL^2_x([0,\infty)\times\{\langle x\rangle\approx 2^j\})}\right)\lesssim E[u](0),$$

- Set

$$\|u\|_{LE}:=\sup_{j\geq 0}\|\langle x\rangle^{-1/2}u\|_{L^2_{t,x}([0,\infty)\times\{\langle x\rangle\approx 2^j\})}$$

$$\|u\|_{LE^1}:=\|\partial u\|_{LE}+\|\langle x\rangle^{-1}u\|_{LE}$$

$$\|u\|_{LE^*}:=\sum_{j=0}^{\infty}\|\langle x\rangle^{1/2}u\|_{L^2_{t,x}([0,\infty)\times\{\langle x\rangle\approx 2^j\})}$$

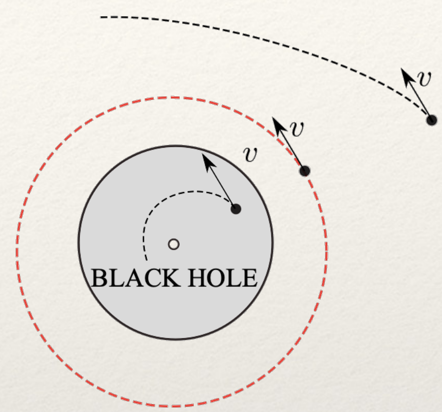
- Above estimate becomes $\|u\|^2_{LE^1}\lesssim E[u](0)$

- Local energy decay (LED) estimate:

$$\|u\|_{LE^1}+\|\partial u\|_{L^\infty_tL^2_x}\lesssim\|\partial u(0)\|_{L^2}+\|f\|_{LE^*+L^1_tL^2_x}.$$

Obstructions to LED

1. Trapping: null geodesics stay in a compact set forever
   - Example: black holes
   - Trapping implies no LED
   - Can recover weaker LED statements for certain types of trapping



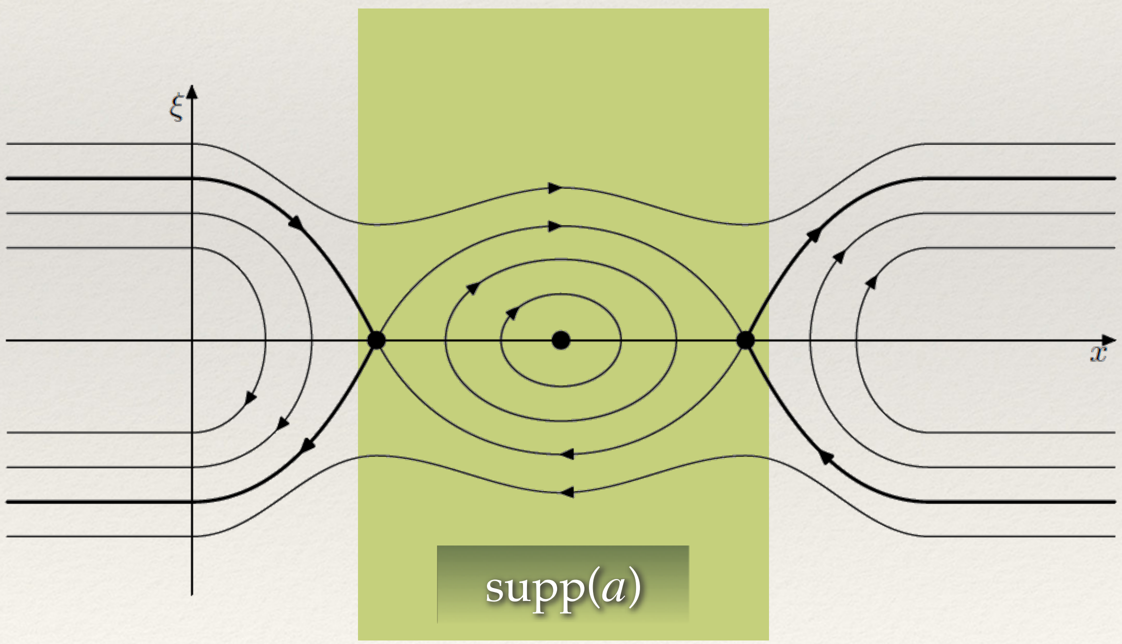
2. Bad spectrum
   - Negative eigenfunctions (yield exponential growth in time)
   - Real resonances (more subtle growth)
     - Non-zero resonances/embedded eigenvalues (don't occur for symmetric operators)
     - Zero resonance/eigenvalue

Related problem: Can we recover LED if we use damping to control the trapping?

Damped Waves and Geometric Control

- Geometric control condition (GCC): every bounded null geodesic intersects $\{a > 0\}$
- First introduced by Rauch and Taylor ('75) to establish exponential decay for hyperbolic equations on compact product manifolds without boundary

$$E[u](t) \lesssim e^{-\alpha t} E[u](0), \qquad \alpha > 0$$



Example of geometric control

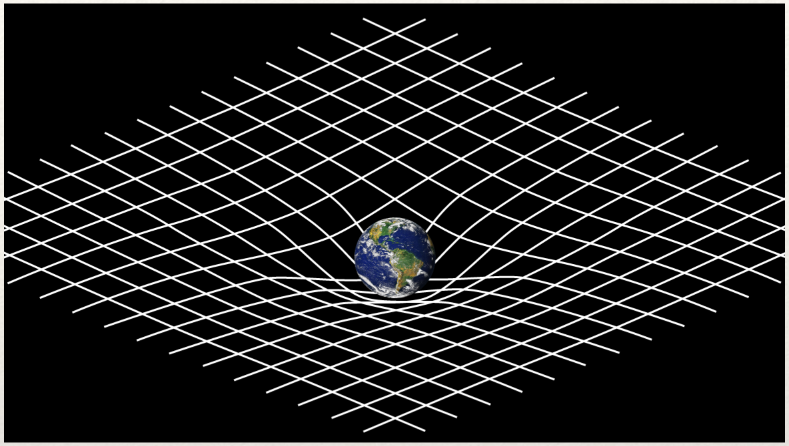
Adding in Space-Time Geometry

Add in space-time geometry

- $(\mathbb{R}^4, g)$ Lorentzian, $\mathrm{sgn}(g) = (-+++)$
- Damped wave operator

$$P = D_\alpha g^{\alpha\beta} D_\beta + iaD_t$$

$a \in C_c^\infty(\mathbb{R}^3)$ non-negative and positive on an open set



- Assumptions: g asymptotically flat, P stationary
- Bouclet and Royer ('14) showed LED for damped waves on asymptotically Euclidean (i.e. product) manifolds satisfying GCC
  - Metrics of the form $-dt^2 + g_{ij}dx^i \otimes dx^j$
  - Induce damped wave operators $-D_t^2 + D_i g^{ij} D_j + ia(x)D_t$
- Famous example of metric with non-product structure: Kerr space-time

Main Theorem: LED

Theorem: Suppose that P is a stationary, asymptotically flat damped wave operator satisfying the geometric control condition, and let ∂_t be uniformly time-like. Then, we have the estimate

$$\|u\|_{LE^1[0,T]} + \|\partial u\|_{L^\infty_t L^2_x[0,T]} \lesssim \|\partial u(0)\|_{L^2} + \|Pu\|_{LE^* + L^1_t L^2_x[0,T]},$$

and the implicit constant is independent of T.

Will follow outline of Metcalfe, Sterbenz, and Tataru ('20) (waves on non-trapping, Lorentzian space-times); modifications via Bouclet and Royer arguments where trapping takes effect

General strategy:

1. Prove high, medium, and low frequency estimates
2. Combine together to prove that

$$\|u\|_{LE^1} \lesssim \|Pu\|_{LE^*}, \qquad u \in \mathcal{S}(\mathbb{R}^4)$$

3. Prove that this implies LED by constructing a function which matches the Cauchy data of u at times 0 and T.

High Frequency Estimate

- Big idea: want an estimate that implies LED for time frequencies in a neighborhood of infinity
- Trapping is high frequency
- **Theorem**: Suppose that P is a stationary, asymptotically flat damped wave operator satisfying the geometric control condition, and let ∂_t be uniformly time-like. Then, we have the estimate

$$\|u\|_{LE^1[0,T]} + \|\partial u\|_{L^\infty_t L^2_x[0,T]} \lesssim \|\partial u(0)\|_{L^2} + \|\langle x\rangle^{-2} u\|_{LE[0,T]} + \|Pu\|_{LE^* + L^1_t L^2_x[0,T]},$$

and the constant is independent of T.

Proof Idea I

- Want to use positive commutator argument and pseudodifferential calculus to complete proof
  - For self-adjoint P
    $$2i\mathrm{Im}\langle Pu, Qu\rangle = \langle [P,Q]u, u\rangle$$
  - Want Q so that quadratic form is positive up to lower-order errors
    - Via microlocal methods, want a symbol q so that $\{p, q\} > 0$
  - Damping is skew-adjoint: generate anti-commutators which turn into function multiplication via microlocal analysis
  - So, we want symbols which give appropriate positivity
- Just like for flat LED, we need to construct a primary symbol (escape function) and a correction symbol

Proof Idea II

- Construct symbols in steps:

  1) On the characteristic set

     I) Interior ($\{|x| \leq R\}$) points along semi-bounded geodesics $\leftarrow$ this is where geometric control is used

     II) Remainder of interior region

     III) Exterior region $\{|x| > R\}$

  2) On the elliptic set

Low Frequency Estimate

- Big idea: want estimate that implies LED in a neighborhood of zero frequency
- Let $P_0 = P\Big|_{D_t=0} = D_i g^{ij} D_j$
- ∂_t uniformly time-like $\Longrightarrow$ P_0 uniformly elliptic $\Longrightarrow$ no resonance at zero
- **Theorem**: Suppose that P is an asymptotically flat damped wave operator, and let ∂_t be uniformly time-like. Then, we have the estimate

$$\|u\|_{LE^1} \lesssim \|\partial_t u\|_{LE^1_c} + \|Pu\|_{LE^*}.$$

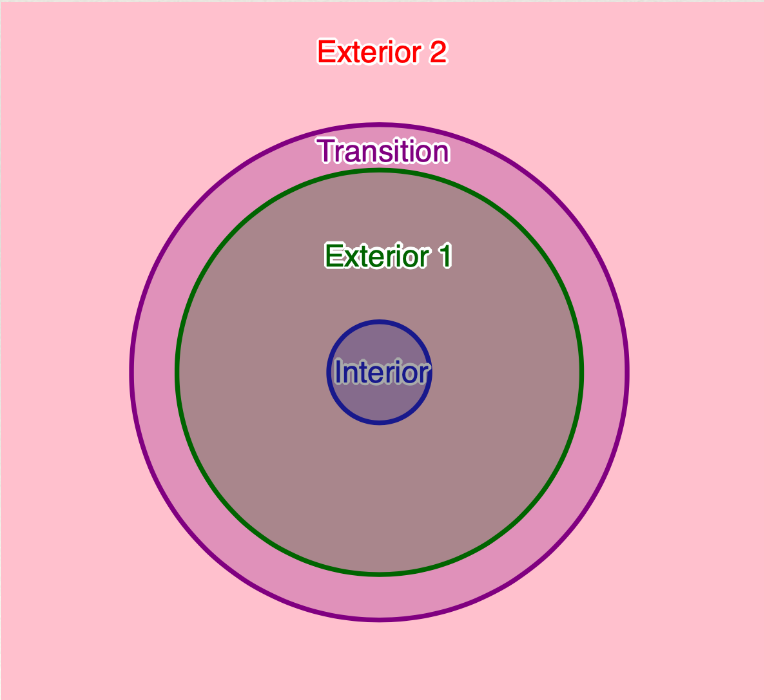
- Proof idea
  - Establish weighted estimates for Δ
  - Use perturbation arguments to get weighted estimates for P_0
  - Integrate in time

Medium Frequency Estimate

- Big idea: want estimate that implies LED for any range of time frequencies bounded away from zero and infinity
- Built on Carleman estimates

$$\|\rho_0 e^{\varphi} u\|_{L^2_t L^2_x} + \|\rho_1 e^{\varphi} \partial u\|_{L^2_t L^2_x} \lesssim \|e^{\varphi} P u\|_{L^2_t L^2_x}$$

- Overall proof idea
  - Get Carleman estimate outside of compact set
    - Bend weight to apply exterior estimate
  - Get Carleman estimate within compact set
  - Combine together
- Proof idea for a Carleman estimate: Positive commutator arguments on conjugated operator $P_\varphi = e^{\varphi} P e^{-\varphi}$



Spectral Theory

- Another approach: Spectral theory
- Define $\mathscr{L}\mathscr{E}, \mathscr{L}\mathscr{E}^1$ by removing time dependence, and

$$\mathscr{L}\mathscr{E}^1_\omega = \mathscr{L}\mathscr{E} \cap |\omega|^{-1} \mathscr{L}\mathscr{E}^2$$

$$\dot{H}^1_\omega = \dot{H}^1 \cap |\omega|^{-1} L^2$$

- Consider $Pu = 0, \qquad u(0) = 0, \qquad -g^{00}\partial_t u(0) = f$
- Using Fourier-Laplace transform, define $R_\omega f = \int_0^\infty u(t) e^{i\omega t}\, dt$
- Using energy coercivity, get uniform resolvent bound

$$\|R_\omega f\|_{\dot{H}^1_\omega} \lesssim |\operatorname{Im} \omega|^{-1} \|f\|_{L^2}, \qquad \{\operatorname{Im} \omega < 0\}$$

- LED holds iff local energy resolvent bound holds:

$$\|R_\omega f\|_{\mathscr{L}\mathscr{E}^1_\omega} \lesssim \|f\|_{\mathscr{L}\mathscr{E}^*}, \qquad \{\operatorname{Im} \omega < 0\}$$

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